Last time – reduction of order

Lets hear presentations from our three remaining groups!

Quick Quiz – Reading! – please keep up!

Another Example – Just in notes – not in class:

$$\dddot{x} + b\dot{x} + cx = \sin(x^3)$$

Third order nonlinear ode.

define new variables: $x_1 \quad x_2 \quad x_3$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3 \qquad \text{so} \quad \dot{x}_3 = \ddot{x}_2 = \dddot{x}_1$$

I need the third derivative

$$\dddot{x} = \sin(x^3) - b\dot{x} - cx$$

so $\left\{ \begin{array}{l} x = x_1 \\ \dot{x} = x_2 \\ \ddot{x} = x_3 \end{array} \right.$

$$\underbrace{\dddot{x}_1 = \dot{x}_3}_{} = \underbrace{\sin(x_1^3) - bx_2 - cx_1}_{}$$

plug in from here

New system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = \sin(x_1^3) - bx_2 - cx_1$$

Double check: Third order ode – I should see THREE first order equations

Only first derivatives on Left Side

No derivatives only x_1, x_2, x_3 on Right Side.

Today – We Start <u>Flows on a Line</u>

pages 15 – 24 in the book.

our work in class follows this closely.

Flow on a Line:

In this class we will be analyzing very complicated systems:

$$\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, x_2, \ldots, x_n)$$

- we think of solutions as trajectories flowing through n-dimensional space with coordinates $(x_1, x_2, x_3, \ldots, x_n)$ which change in time.

This idea is called PHASE SPACE
can't always be graphed... > 3 dimensions.

Start in 1-dimension and build up our reasoning from there.

$\dot{x} = f(x)$ one dimensional system
autonomous ode.

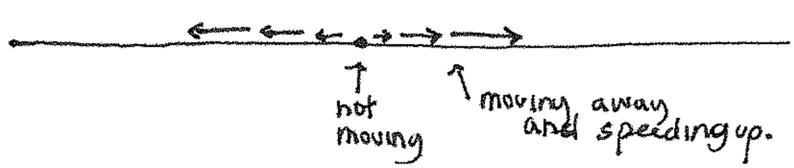
Autonomous: The independent variable does <u>not</u> appear in our equation.

if we had $\dot{x} = f(x,t)$
we would think of this as two dimensions since the phase space changes in time!

$$\begin{cases} \text{let } \dot{x}_1 = f(x_1, x_2) \\ \text{and } x_2 = t \text{ so that} \\ \dot{x}_2 = 1 \end{cases}$$

We imagine phase space as a vector field. Every point in space has a vector that tells your solution what direction it is going and how fast.

In 1-dimension — this is a Flow on a Line.



To demonstrate a big example!

Lets consider the equation:

$$\dot{x} = \sin(x), \quad x(0) = x_0$$

This is seperable and can be solved:

$$\frac{dx}{dt} = \sin(x)$$

$$\int \frac{dx}{\sin(x)} = \int dt$$

$$\ln\left|\frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)}\right| = t$$

a big mess... to get general soln apply initial condition

this is an exact result – but what does it tell us?

1. If $x_0 = \pi/4$ then what does the solution do for $t > 0$?

2. What happens as $t \to \infty$?

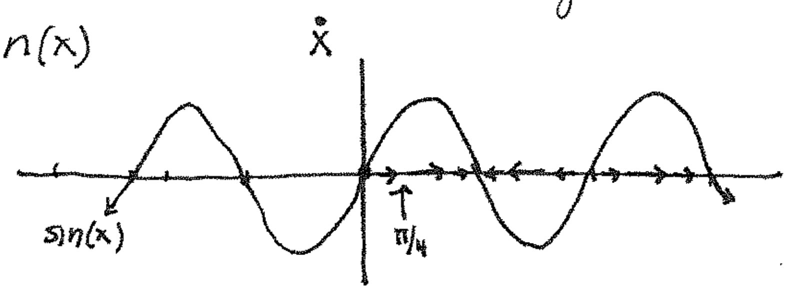
Hard to say given how messy our solution is!

<u>Instead</u>

Think of the solution as a flow on a line:

PLOT $\dot{x}$ vs $f(x)$ to find how the velocity changes at each position.

$\dot{x} = \sin(x)$



#1 plot $f(x)$

#2 Draw vectors

FIXED POINTS – points where $\dot{x} = 0$, no change if the solution is at this point it does not move!

STABLE F.P. – solutions go toward stable fixed points

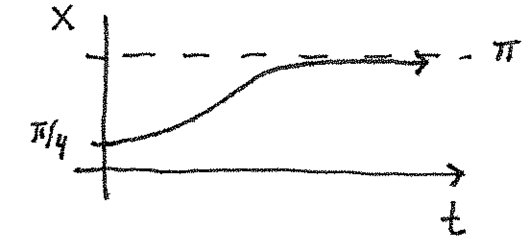
UNSTABLE F.P. – solutions go away from unstable fixed points.

The flow on the line lets us answer our questions!

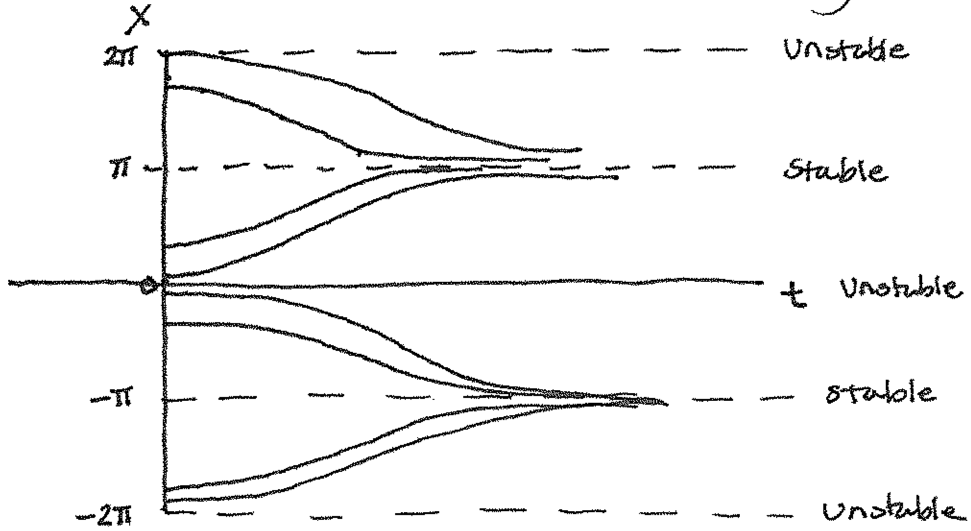
1. $x_0 = \pi/4$ what will happen...

It will move to the right, or get larger, also moving faster until $\pi/2$ when it will start slowing down and approaching $x=\pi$. As $t \to \infty$ the solution settles down at $x=\pi$.

SKETCH THIS



2. We could do the same for any initial condition.



Each of these lines is a TRAJECTORY.
they correspond to different initial conditions.

PHASE PORTRIATE – plot of a range of possible trajectories.

FIXED POINTS · where $\dot{x}=0$ also called EQULIBRIUM POINTS.

Other Questions:

* When is The system moving the fastest?
* when should we expect increasing (or decreasing) solutions?

GROUP WORK...

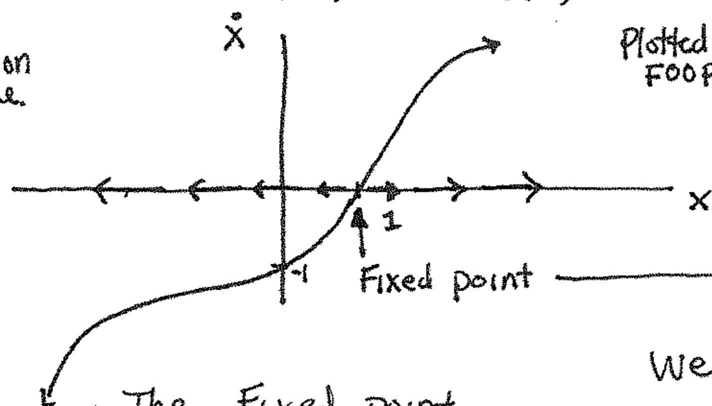
Another Example — Just in notes — not in class.

Consider $\dot{x} = x - \cos(x)$
determine the behavior of this system.

Fixed Points, Stability, Phase Diagram
Description of the solution.

Plot $\dot{x}$ vs $f(x) = x - \cos(x)$

Flow on Line.



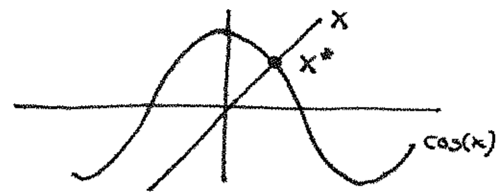
Plotted on FOOPLOT.

Fixed point → Note this is where $\dot{x} = 0$ or $f(x) = 0$

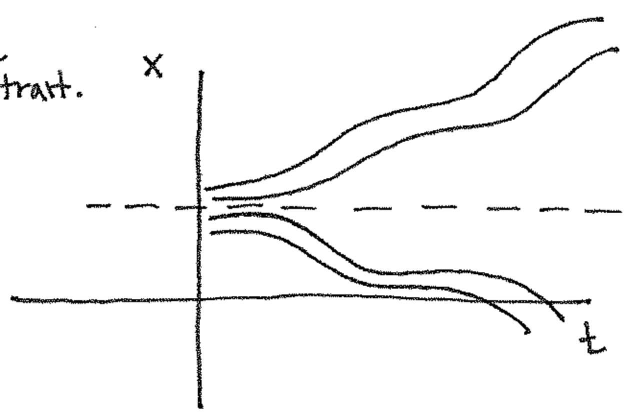
The Fixed point is unstable.
On either side the solution goes away!

We can solve for this!
$x^* - \cos(x^*) = 0$

$x^* = \cos(x^*)$



Phase Portrait.



x^* unstable → Note we could also plug in points to the left and right $x^* \pm \epsilon$ to test positive or negative flow.

Describe — If we start above x^* then the solution will increase going to ∞ as $t \to \infty$.
the velocity will always be positive but the magnitude will oscilate.

The opposite happens if we start below x^*
the solution goes to $-\infty$ as $t \to \infty$.

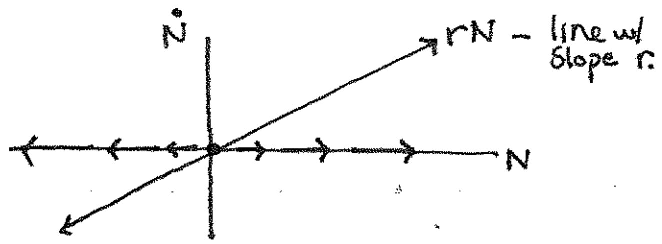
Population Dynamics...

SIMPLE MODEL vs. LOGISTIC GROWTH

$\dot{N} = rN$ $\qquad$ r = growth rate > 0 $\qquad$ $\dot{N} = rN(1 - N/K)$

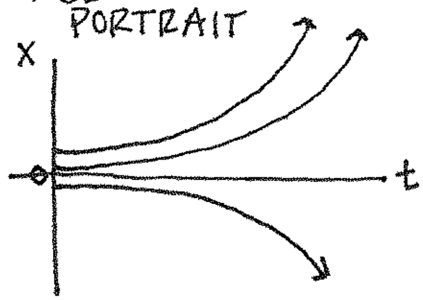
K = carrying capacity.

FLOW ON LINE

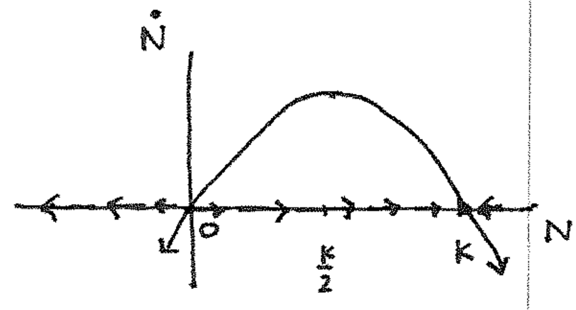


Fixed point at $N^* = 0$ unstable.
The velocities get larger in magnitude as we get further from the fixed point.

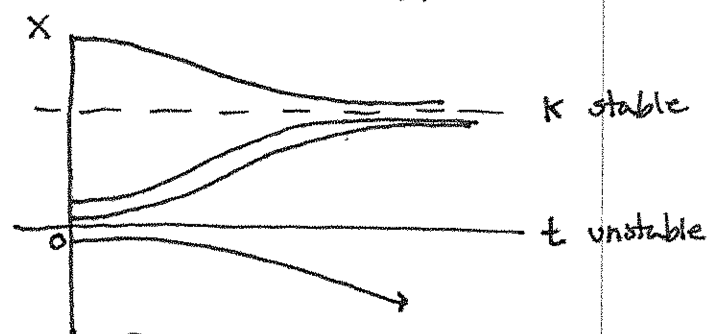
PHASE PORTRAIT



Population just grows exponentially

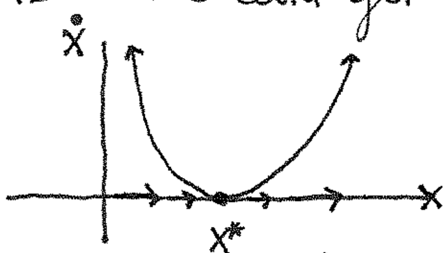


Fixed points at $N^* = 0$ and $N^* = K$ one stable the other unstable.



Population slowly approaches the carrying capacity

NOTE — We could get a BISTABLE fixed point.



here if we start with $X < X^*$ then our solution approaches X^*

if we start with $X > X^*$ our solution goes away from X^*