

Change of Variables for ODE's.

How is the homework going?
You should be done
Thru population dynamics

Given an ode — sometimes there is a really obvious way to simplify using substitution.

$$\dot{x} = (2x + t + 3)^2$$

first order
non-linear
not separable.

similar to integration...
I see a function inside that I would like to replace.

let $V = 2x + t + 3$

we are changing from $x(t)$ to $V(t)$.

1. Choose

the substitution.

2. Take derivative

$$\dot{V} = 2\dot{x} + 1$$

use the chain rule

$$\frac{d}{dx}(2x) = 2 \frac{dx}{dt}$$

$$\text{so } \dot{x} = \frac{1}{2}[\dot{V} - 1]$$

3. Sub into the original eqn:

$$\frac{1}{2}[\dot{V} - 1] = V^2 \quad \text{and rearranging}$$

$$\dot{V} = 2V^2 + 1$$

this is separable.

4 Solve

$$\frac{dV}{dt} = 2V^2 + 1$$

$$\int \frac{dV}{2V^2 + 1} = \int dt$$

$$\frac{1}{\sqrt{2}} \arctan(\sqrt{2}V) = t + C$$

$$\text{or } V = \frac{1}{\sqrt{2}} \tan(\sqrt{2}t + C)$$

5 return to original vars.

$$2x + t + 3 = \frac{1}{\sqrt{2}} \tan(\sqrt{2}t + C)$$

$$x = \frac{1}{2} \left[\frac{1}{\sqrt{2}} \tan(\sqrt{2}t + C) - t - 3 \right]$$

Last time we talked about a graphical analysis for fixed points and stability.

Today we develop the formal mathematics

Fixed points are points where the system is not changing.

given $\dot{x} = f(x)$ the fixed point x^* is where $f(x^*) = 0$.

To test the stability of the fixed point we look just to the left and just to the right to see if the solution returns or moves away from the fixed point.

PERTURBATION ANALYSIS — find a steady state or fixed point or equilibrium point...

Then "perturb" the solution a bit to see what it does and to determine the stability.

Given $\dot{x} = f(x)$

Imagine we have our fixed point $f(x^*) = 0$



Now look just to the ~~left~~ right:

(*) $x = x^* + n$ give the solution a push (arbitrary) in pos. direction

Then ask what happens to this small push

$n = x - x^*$ "does it increase or decrease"

How DOES n change

→ Take the derivative

$$\dot{n} = \frac{d}{dt}(x - x^*) = \dot{x} = f(x) = f(x^* + n)$$

just subbing in (*)

We will look at the LINEAR STABILITY of $\dot{n} = f(x^* + n)$

Remember: In general $f(x^*+n)$ is a nonlinear function. We want to find a linear approximation to it.

TAYLOR SERIES!

We can expand a function about a point as a series of polynomials.

$$f(x) \underset{x \text{ near } a}{=} f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \quad \text{HOT}$$

$$f(x^*+n) = f(x) \underset{\substack{\text{lets expand} \\ \text{near } x=x^*}}{=} f(x^*) + f'(x^*)(x-x^*) + \dots \quad \text{HOT}$$

Simplify $x-x^* = n$
defined above.

and $f(x^*) = 0$ this is the definition of a fixed point.

so

$$f(x^*+n) = f'(x^*)n + \dots \quad \text{HOT}$$

our Linear approximation

the next term contains n^2 and if n is small n^2 is even smaller.

$$f(x^*+n) \sim f'(x^*)n$$

Recall we are asking how does n change so

assuming $f'(x^*) \neq 0$
if $f'(x^*) = 0$ linear stability tells us nothing.

$$\dot{n} = f(x^*+n) \sim n f'(x^*) \quad \text{slope determines stability!}$$

IF our "push" grows then $\dot{n} > 0$ and

$f'(x^*) > 0$ means unstable.

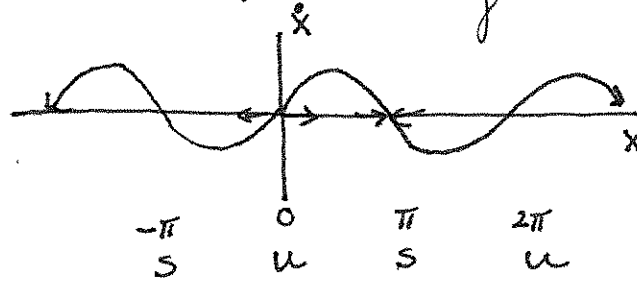
IF our "push" decays then $\dot{n} < 0$ and

$f'(x^*) < 0$ means stable.

The magnitude of $f'(x^*)$ estimates how stable the fixed point is.

Ex Consider $\dot{x} = \sin(x)$ from yesterday

Flow on a line



Plot of velocity at each position.

LINEAR STABILITY $\rightarrow$

FIXED POINTS

$$f(x^*) = \sin(x^*) = 0$$

$$x^* = 0, \pm\pi, \pm 2\pi, \dots$$

$$x^* = n\pi \quad n = \text{integer}$$

STABILITY. consider $f'(x^*) = \cos(x^*) = \cos(n\pi) = (-1)^n$

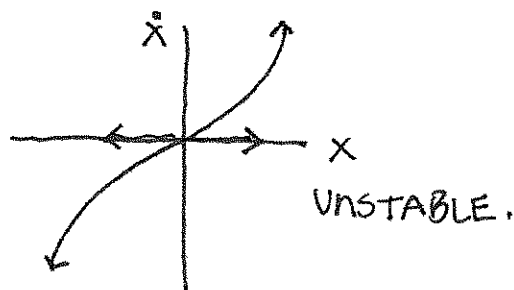
$$\begin{aligned} n=0 & \quad f'(0) = 1 > 0 \quad \text{so } x^*=0 \text{ is unstable} \\ n=1 & \quad f'(\pi) = (-1)^1 = -1 < 0 \quad \text{so } x^*=\pi \text{ is stable.} \end{aligned}$$

$$x^* = n\pi \text{ is } \begin{cases} \text{stable} & n = \text{odd} \\ \text{unstable} & n = \text{even (zero)} \end{cases}$$

Also $|f'(x^*)| = 1$ so all points are equally stable or unstable.

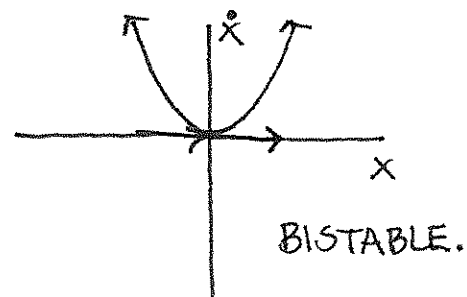
Cases where $f'(x^*) = 0$

$$\begin{aligned} \dot{x} &= x^3 \\ x^* &= 0 \\ f'(x^*) &= 3(x^*)^2 = 0 \end{aligned}$$



AND

$$\begin{aligned} \dot{x} &= x^2 \\ x^* &= 0 \\ f'(x^*) &= 2x^* = 0 \end{aligned}$$



NOTE: ○ UNSTABLE

● STABLE

● BISTABLE.

Existence and Uniqueness — nonlinear problems often have existence and uniqueness problems.

Existence Problem → There is no solution.
The problem is ill posed.
Not even a numerical solution will help.

Uniqueness Problem → There is more than one solution for the given initial condition.
Can't mathematically choose just one.

Consider the initial value problem
 $\dot{x} = F(x, t) \quad x(t_0) = x_0$

Existence:

Suppose that $F(x, t)$ is a continuous function defined on some open interval R^* containing the point (x_0, t_0) . Then the initial value problem has a solution on some interval about $t = t_0$.

Aside — From Reals:

* $R = \{ (x, t) : t_0 - \delta < t < t_0 + \delta, x_0 - \epsilon < x < x_0 + \epsilon \}$
containing the point (x_0, t_0) . Then there exists a δ_1 (possibly smaller than δ) so that a soln $x = f(t)$ is defined for $t_0 - \delta_1 < t < t_0 + \delta_1$.

SIMPLY — Ask if the function $F(x, t)$ is continuous near the point (x_0, t_0) of the initial condition.
If so a solution exists.

Uniqueness:

Suppose that both $F(x, t)$ and $\frac{\partial F}{\partial x}(x, t)$ are continuous on the region R , containing the point (x_0, t_0) . Then the initial value problem has a unique solution on some interval about $t = t_0$.

Aside — From Reals

* Then there exists a δ_2 (possibly smaller than δ_1) so that the solution $x = f(t)$, whose existence is guaranteed above, is the unique solution for $t_0 - \delta_2 < t < t_0 + \delta_2$.

SIMPLY — If both $F(x, t)$ and $\frac{\partial F}{\partial x}(x, t)$ are continuous the the soln exists and is unique.

Ex $\dot{x} = 1 + x^2$
 $x(0) = x_0$ $\left. \begin{array}{l} f(x) = 1 + x^2 \\ f'(x) = 2x \end{array} \right\}$ solutions exist and are unique.

solve this ... $\int \frac{dx}{1+x^2} = \int dt$

$$\arctan(x) = t + c$$

$$x(t) = \tan(t + c)$$

and

$$x(0) = \tan(c) = x_0$$

$$c = \arctan(x_0)$$

let $x_0 = 0$ then $c = 0$

$$x(t) = \tan(t)$$

Solutions exist and we ~~for~~ found the unique soln.

Even nice solutions still have problems! ..

Here $x(t) \rightarrow \infty$ as $t \rightarrow \pm \pi/2$

infinite solutions in finite time!

EXAMPLE of BLOW UP



Work in groups of
2-3 and figure
these out too

In Class Thought Experiments

1. (The leaky bucket) The following example (from Hubbard and West 1991, p.159) shows that in some physical situations it is ok to have a non-unique solution. Non-uniqueness is natural and obvious not pathological.

Consider a water bucket with a hole in the bottom. You are walking down the street and you see an empty bucket with a puddle beneath it, can you figure out when the bucket was full? Why or Why not?

Let's consider a differential equation that describes water leaking from a bucket:

$$\dot{h} = -C\sqrt{h}$$

where $C = \sqrt{2g} \frac{a}{A}$. h represents the height of water in the bucket, g is the gravitational constant, a is the area of the hole in the bottom of the bucket, and A is the cross-sectional area of the bucket. Solve this equation for the case of $h(0) = 0$, at $t = 0$ when we found the bucket it was empty. You should find two solutions that work here. What are the two possibilities? What do they mean physically, as we go backwards in time? Try to explain how these solutions make sense.

2. Explain this paradox: a simple harmonic oscillator $m\ddot{x} = -kx$ is a system that oscillates in one dimension (along the x-axis). But we just said that one-dimensional systems can't oscillate, how can this be true?

6.
In General we will deal w/ NICE functions!

2. Impossibility of Oscillations.

Fixed points dominate the dynamics of first order systems:

ALL TRAJECTORIES EITHER APPROACH

A FIXED POINT OR DIVERGE to $\pm\infty$!!!

These are the ONLY thing that can happen for a first order system.

Since the solution doesn't overshoot and the phase point never reverse direction $\rightarrow$ NO OSCILLATIONS!

- These results come from topology. If we think of $\dot{x} = f(x)$ as a flow on a line then you can never return to your starting point
- We would need a flow on a circle or in 2-D to return to where we started.