Homework due Sunday 12 Noon
Dropbox!

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Today: { Another way to visualize first order equations: POTENTIALS
Numerical Analysis for first order systems.

Where are we now...

Given a first order system $\dot{x} = f(x)$
we can do quite an analysis.

1. Check for existence or uniqueness

2. Get information from the system:

- Find fixed points / Equilibrium Points
- Sketch vector field on the line
- Discuss stability of fixed points
- Linear Stability Analysis
- Draw typical Solution Curves.

3. In some cases solve the system... if we can integrate $\frac{1}{f(x)}$.

Potentials - give another way to get info from the system

Numerical Analysis - gives us a way to solve problems that we cant integrate.

- Slope fields
- Numerical Integration

REMEMBER: #1 goal - creative problem solving... even in your other classes you can use these methods.

Potentials – another way to visualize ODE's.

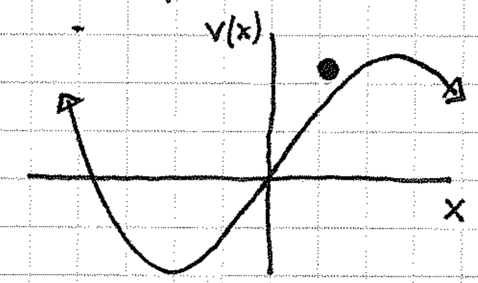
given $\dot{x} = f(x)$

we can visualize a particle sliding down the walls of a potential well where the potential $V(x)$ is:

$$f(x) = -\frac{dV}{dx}$$ $\longrightarrow$ negative sign $\Rightarrow$ particle will move downhill

How does this work: 1 we solve for $V(x)$ by integrating.

2. We plot $V(x)$ vs. x



3. Imagine a particle moving or rolling down the potential.
The particle is very sticky or is slogging through goo!

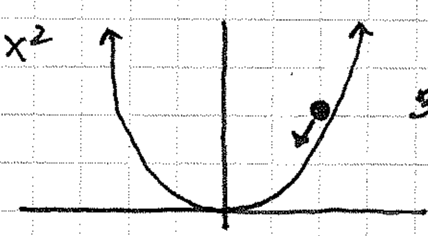
4. Then we can discuss the stability of fixed points based on this picture

Ex $\dot{x} = -x$ has fixed point @ $x=0$

1. $-\frac{dV}{dx} = -x \qquad \frac{dV}{dx} = x \qquad V = \frac{1}{2}x^2 + C$

when dealing w/ potentials $C=0$ (always. arbitrary)

2. Plot $V = \frac{1}{2}x^2$



3. Imagine a particle... It will go down the walls to $x=0$

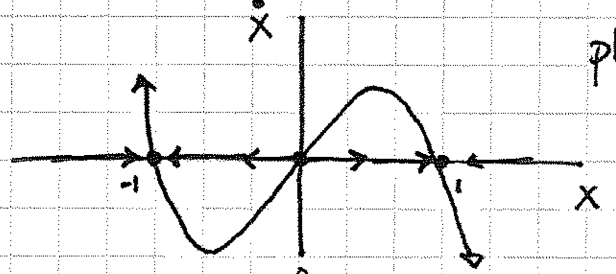
4. So $x=0$ is a stable equilibrium.

Ex $\dot{x} = x - x^3$

fixed points: $x(1-x^2)=0$ $x=0$ and $x=\pm 1$

FLOW ON A LINE

plot $\dot{x}$ vs x



stable, unstable, stable

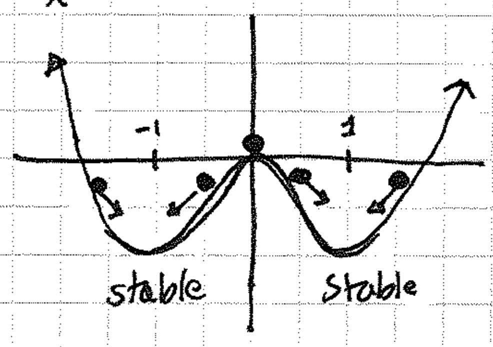
POTENTIAL $-\dfrac{dV}{dx} = x - x^3$

$$V = \int x^3 - x \, dx$$

$$= \tfrac{1}{4}x^4 - \tfrac{1}{2}x^2 + c$$

$$= 1/4 \, x^4 - 1/2 \, x^2$$

plot $V(x)$ vs x



$x^4 - 2x^2 = 0$

$x^2(x^2-2) = 0$

$x=0$ $x=\pm\sqrt{2}$

Double well potential...

stable Stable

unstable

call this bi-stable because it has two stable equilibrium points.

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Solving Equations on the Computer: (We will discuss three methods)

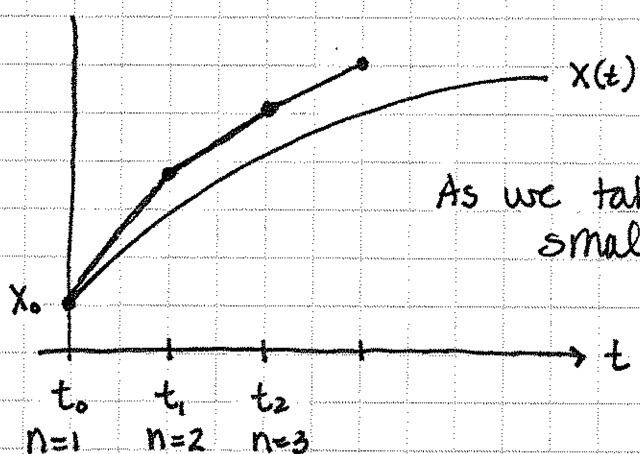
$$\dot{x} = f(x) \quad , \quad x(t_0) = x_0$$

Eulers Method: "Discretize the Eqn"

$$x(n+1) = x(n) + \Delta t\, f(x(n))$$

can be derived from the definition of 1^{st} derivative $\frac{dx}{dt} \sim \frac{x(n+1) - x(n)}{\Delta t}$.

our solution at the next discrete time step = where we were at the last step + change in time · The SLOPE of velocity at last step.



As we take smaller and smaller steps the solution gets better

NOTE: Eulers method gets very bad approximations unless Δt is very small!

Ex $\dot{x} = \sin(x)$

#1 do work by hand

to discretize this we plug into the formula ... I always do this part explicitly by hand!

$$x(n+1) = x(n) + \Delta t f(x(n))$$

so really just plug in for $\sin(x)$

$$x(n+1) = x(n) + \Delta t \cdot \sin(x(n))$$

#2 Then program the computer

Then use a computer to itterate.

Put Psuedo Code on Side Board.

Improved Euler – euler method only uses info from the left side of the time step... this causes a lot of error.

Improved Euler: Calculate the Slope by Averaging the slope on the right and left!

1. Find the location where the euler method would send you

$$\bar{x}(n+1) = x(n) + \Delta t f(x(n)) \quad \text{like a trial step}$$

2. Calculate the actual step using the average slopes.

$$x(n+1) = x(n) + \Delta t \cdot \tfrac{1}{2}\Big[\underbrace{f(x(n))}_{\text{slope on the left side}} + \underbrace{f(\bar{x}(n+1))}_{\text{estimated slope on the right side.}}\Big]$$

This method has smaller error $E = |\underbrace{x(t_n)}_{\text{real soln}} - \underbrace{x(n)}_{\text{estimated soln.}}|$ than Euler.

For Both methods as $\Delta t \to 0 \quad E \to 0$
But $E \to 0$ faster in the Improved Euler Method

Euler Method $\quad E \propto \Delta t \quad$ called an order one method

Improved Euler $\quad E \propto (\Delta t)^2 \quad$ called an order two method

EX $\quad \dot{x} = \sin(x)$

$$\bar{x}(n+1) = x(n) + \Delta t \sin(x(n)) = x_{trial}$$

so...

$$x(n+1) = x(n) + \Delta t \cdot \tfrac{1}{2}\left[\sin(x(n)) + \sin(x_{trial})\right]$$

FOURTH ORDER RUNGE-KUTTA METHOD: Idea we calculate the change in x a few different ways then take a weighted average to give a best guess.

1. Calculate four Distance estimates:

$k_1 = f(x(n))\Delta t$ → like the regular euler estimate left.
$k_2 = f(x(n) + \frac{1}{2}k_1)\Delta t$ → distance using slope in middle
$k_3 = f(x(n) + \frac{1}{2}k_2)\Delta t$ → distance using middle slope
$k_4 = f(x(n) + k_3)\Delta t$ → distance using middle slope

2. Put these all together

$$x(n+1) = x(n) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

This is a forurth order method $E \propto (\Delta t)^4$
so it does not require small step sizes to get accurate results.

For more information TAKE NUMERICAL ANALYSIS!

Q: Why don't we just use the easiest method and choose a SUPER small Δt and let the coputer run?

* ROUND-OFF ERROR – computers don't have infinte accuracy – they dont distinguish between numbers that are very close together

$\delta \equiv 10^{-7}$ for single precision
$\delta \equiv 10^{-16}$ for double precision
} if Δt is too small these will really add up!

* TAKES TOO LONG.

Ex $\dot{x} = \sin(x)$

$$k_1 = \sin(x(n))\,\Delta t$$
$$k_2 = \sin(x(n) + \tfrac{1}{2}k_1)\,\Delta t$$
$$k_3 = \sin(x(n) + \tfrac{1}{2}k_2)\,\Delta t$$
$$k_4 = \sin(x(n) + k_3)\,\Delta t$$

NOTE ... all the arguments are inside $\sin(x)$ we take the sine of $x(n) + \tfrac{1}{2}k_1$ – this is a #.

Then
$$x(n+1) = x(n) + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

For any of these methods ... PSUDEO CODE:

Do this first
- input start time and endtime and Δt
- input initial condition $x_0 = x(1)$
- decide how many steps to take $N = \dfrac{\text{endtime} - \text{starttime}}{\Delta t}$

for $n = 1:N$ loop through time steps

(PUT THE DISCRETIZED EQNS HERE)
$$x(n+1) = x(n) + \Delta t \sin(x(n))$$

end

★ This gives a matrix $x(n)$ that contains $N+1$ values for your estimated solution.
Plot your solutions.

ONE FINAL OPTION: SLOPE FIELDS. $\dot{x} = f(x)$ gives the SLOPE of our solution at every point x.

If we ploted all these slopes and then forced our solution ~~too~~ to follow the slope ... we would have a good estimated solution.

In Class Work

Test the numerical methods for the initial value problem $\dot{x} = -x$, $x(0) = 1$

ON THE CLASSROOM COMPUTERS

1. (SLOPE FIELD) Open and run the program dfield8.m in Matlab. This will plot a Phase Portrait for your system. Plot the solution curve corresponding to the initial condition given above.

ON YOUR OWN COMPUTERS

2. (EULER) Open the program euler.m. Go through the code line by line and comment the code saying at each line what the program is doing. Run the program for dt=1, 0.5, 0.1, 0.01. What happens as you decrease your time step?

3. (IMPROVED EULER) Repeat problem 2 for the Improved Euler Method.

4. (RUNGE-KUTTA) Repeat problem 2 for the Runge-Kutta Method.

5. (COMPARE THE METHODS) Run each of the methods for a time step dt=0.1. Which gives the best solution? Which gives the worst?

Numerical Methods
HOMEWORK - Due Sunday Math 11th at Noon.

Now consider the initial value problem $\dot{x} = x + e^{-x}$, $x(0) = 0$. In contrast to the problem from group work it does not have an analytical solution. This means we must use numerical methods to solve the problem and we cannot exactly predict the error. In the code you will not be able to compare to an exact solution.

1a. Use **Matlab on the classroom computers** to plot a slope field and some typical solution curves. Does this equation have fixed points? Do you see any fixed points on your slope field? Save a copy of the slope field graph and put it in your dropbox folder.

1b. On your paper - give a qualitative description of the solution. This should include an analysis of fixed points and stability (Linear Stability Analysis).

2. **On your own computer** use Freemat and the Euler method to find a numerical solution to this equation (You will edit the code we used in class). How small does your time step need to be? (NOTE: remember that if your time step is small enough then your solution will converge and decreasing the time step will not change the solution)

3. Repeat problem 2 using the Improved Euler method.

4. Repeat problem 2 using the Runge-Kutta method.

For each of these problems write up your analytical work, for example analysis of fixed points, a written description of how you discretized the equation for each method, and a discussion of what you found by doing the numerics. Save a plot of your slope fields and numerical solutions. Drop your code (.m files) into dropbox along with your homework solutions and grahps.