Last Time: Potentials $\dot{x} = f(x)$

then $-\frac{dV}{dx} = f(x)$ or $V(x) = -\int f(x)dx$

plot V(x) vs. x and imagine a particle sliding downhill to understand the stability of the system.

– just another way to think of these things.

Numerical Analysis – very important – lots of eqns can't be solved but we need quantitative info!

$\dot{x} = f(x)$

Euler Method:

$$X(n+1) = X(n) + \Delta t f(X(n)) \qquad \text{order } \Delta t.$$

Improved Euler:

$$\bar{X} = X(n) + \Delta t f(X(n))$$

$$X(n+1) = X(n) + \tfrac{1}{2}\Delta t\left[f(X(n)) + f(\bar{X})\right] \qquad \text{order } \Delta t^2.$$

Runge Kutta:

$$k_1 = f(X(n))\Delta t$$

$$k_2 = f(X(n) + \tfrac{1}{2}k_1)\Delta t$$

$$k_3 = f(X(n) + \tfrac{1}{2}k_2)\Delta t \qquad \text{order } \Delta t^4.$$

$$k_4 = f(X(n) + k_3)\Delta t$$

$$X(n+1) = X(n) + \tfrac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

Today – more creative problem solving... Bunny Problem.

Dear Students of Applied Math,

My name is Bongo and I am the C.E.O of a small company that raises rabbits on a large farm. People throughout the USA buy our bunnies as pets, our bunnies are the fluffiest. The problem is that each year we need to take some of the population from our farm to sell, but we are not sure how many bunnies we should take. Some years we take too many, leaving too few to repopulate, and then have to buy more bunnies to start the next year. Other years we take too few, making the farm too crowded, and have to give the bunnies away. Ideally we would keep our population around 100 bunnies but again this varies widely, some years we can't afford to buy more and our population starts as low as 20 bunnies and other years we can't seem to find good homes for the bunnies and our population starts as high as 120.

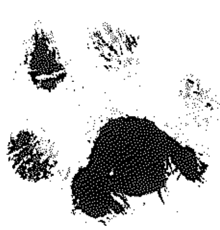
We have had an engineer come study our problem, but he turned out to be a real creep. He took our money and left without giving us any information. The only thing we found in his office was a crumpled up piece of paper with the following written on it:

$$\frac{dP}{dt} = 2P(1 - P/100) - S, \quad P(0) = P_0$$

where t is time measured in weeks, but we don't know what this means. Our company is coming to you for help with this problem. First of all, we would like to understand what this equation means. Second, we would like understand what will happen to our bunny population, depending on how many bunnies we remove each year and how many bunnies we start with. We don't need you to decide how many bunnies should be removed or how many we should start with, we just want a full analysis of the population. Ideally, you could get as much information from this equation as possible for all the different selling rates and starting populations, and summarize it for us. Also, it would be helpful to have the information boiled down to just one or two graphs, so that it is easy to show to the other board members. The group with the winning (nicest, neatest, most informative) graph gets a job with our company!

Thanks in advance for your help!

Bongo Bob



C.E.O.

BIFURCATIONS...

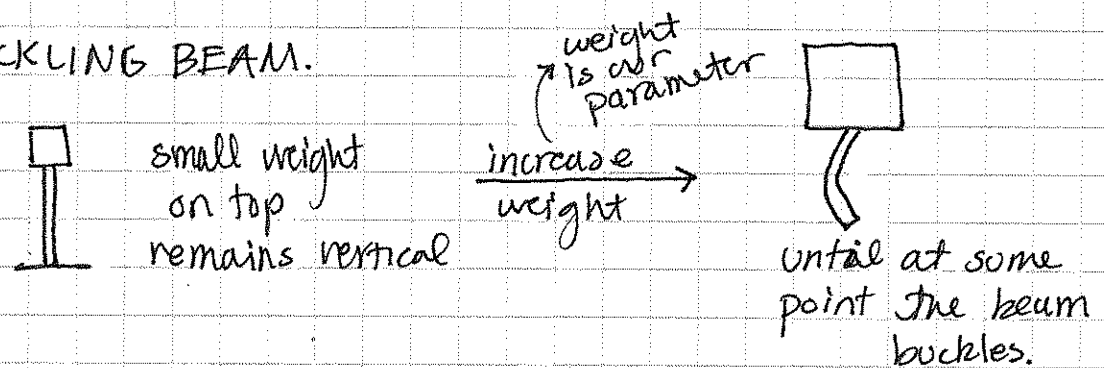
One of the most interesting things about one-dimensional systems is the dependence on parameters...

Sometimes changing just one parameter changes the overall dynamics of the system.

- Qualitative changes in the dynamics: BIFURCATIONS
- Parameter values where they occur: BIFURCATION POINTS.

Why do we care? Scientifically bifurcations provide models of instabilities or transitions.

<u>EX</u> BUCKLING BEAM.



The point where the beam buckles is a bifurcation point for this system.

TYPES OF BIFURCATIONS:
(for 1-D systems)

1. Saddle - Node Bifurcation
2. Transcritical Bifurcation
3. Pitchfork Bifurcation
4. Imperfect Bifurcations + Catastrophes.

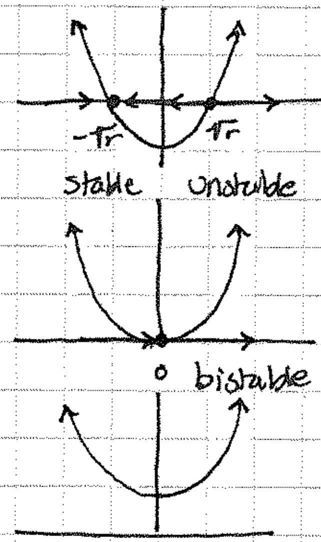
SADDLE-NODE BIFURCATION.

- there are the basic mechanism by which fixed points are created and destroyed.
- as a parameter is varried, two fixed points more toward each other, collide and annihilate.

Ex $\dot{x} = r + x^2$ where r is a parameter.

Fixed points @ $x = \pm\sqrt{-r}$

negative $r \Rightarrow$ two fixed points



$r=0$ only one fixed point

parabola moves upward.

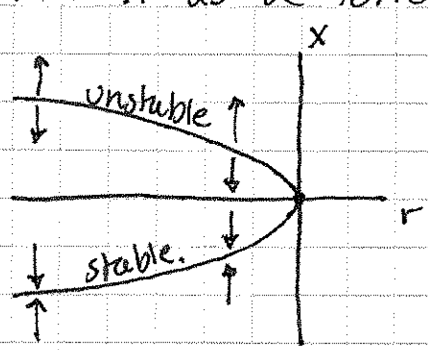
r positive no fixed points

Here we say a bifurcation occurred at $r=0$ because the vector fields for $r<0$ and $r>0$ are qualatatively different.

A BIFURCATION DIAGRAM HELPS VISUALLY DESCRIBE WHAT IS GOING ON!

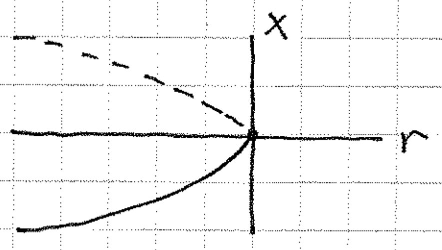
1. Solve $f(x) = 0$ leaving parameter in eqn.
   $r + x^2 = 0$ or $x = \pm\sqrt{-r}$

2. Plot x as a function of r



3. Decide which branch is stable and which is unstable (may have to draw afew vector fields or potentials or do linear stability analysis)

4. Draw full bifurcation diagram.



dashed line – unstable
solid line – stable.

Terminology – lots of different terminology – still a growing subject.

"fold bifurcation" "turning-point"
"blue sky bifurcation"

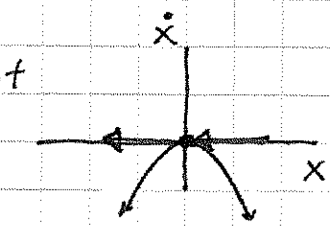
<u>Ex</u> You try: $\dot{x} = r - x^2$

1. Find fixed points
2. Do a linear stability analysis
3. Draw a bifurcation diagram.

1. $f(x) = r - x^2$ so $r - x^2 = 0$ or $x = \pm\sqrt{r}$ are fixed points.

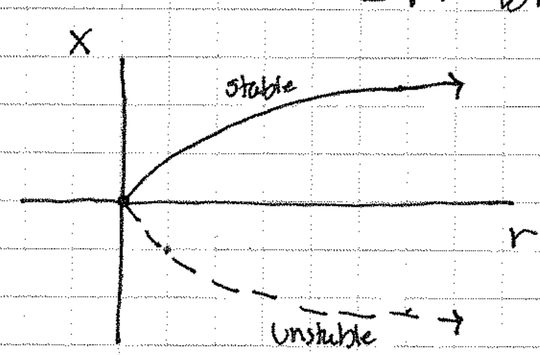
$r =$ negative no fixed points
$r = 0$ one fixed point $x = 0$
$r =$ positive two fixed points $x = \pm\sqrt{r}$

2. $f'(x) = -2x$ $r = 0$ then $x = 0$ is fixed point
$f'(0) = 0$ must draw!
$x = 0$ BISTABLE!



$r =$ positive $f'(\sqrt{r}) = -2\sqrt{r} < 0$ stable
$f'(-\sqrt{r}) = 2\sqrt{r} > 0$ unstable.

3. Diagram: Plot $x(r) = \pm\sqrt{r}$ with $\sqrt{r}$ branch stable
$-\sqrt{r}$ branch unstable



Note we call $\dot{x} = r - x^2$ and $\dot{x} = r + x^2$ NORMAL FORMS.

- They are the prototypical example of saddle-node bifurcations.

A harder example...

<u>Ex</u> $\dot{x} = r - x - e^{-x}$

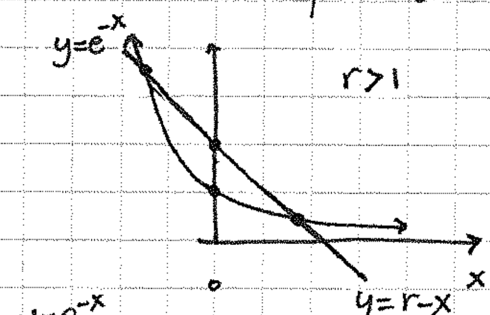
Problem... we can't easily solve for the critical points!

What did we do before? Plot the two functions...

$$r - x - e^{-x} = 0 \quad \text{or} \quad r - x = e^{-x}$$

split into two functions that we can plot.

Now plot different r-values.

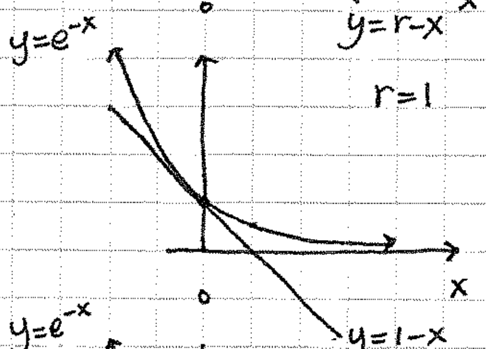


r is the y-intercept
$r-x$ line w/ -1 slope
$y = r - x$ and $y = e^{-x}$
when $x=0$ $e^{-x} = 1$

So when $r>1$ two critical points.

$e^{-x} > r-x$ A $e^{-x} < r-x$ B $e^{-x} > r-x$

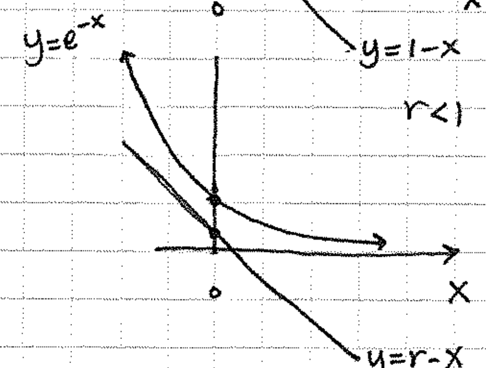
unstable stable



when $r=0$ one critical point

$e^{-x} > r-x$ $e^{-x} > r-x$

Bistable point.



So a bifurcation occurs at $r=1$
or $r=1$ is a bifurcation point.

Two nodes converge to one and annhilate...! Saddle node bifurcation.

TRANSCRITICAL BIFURCATIONS

- some scientific situations require at least one fixed point at all times.

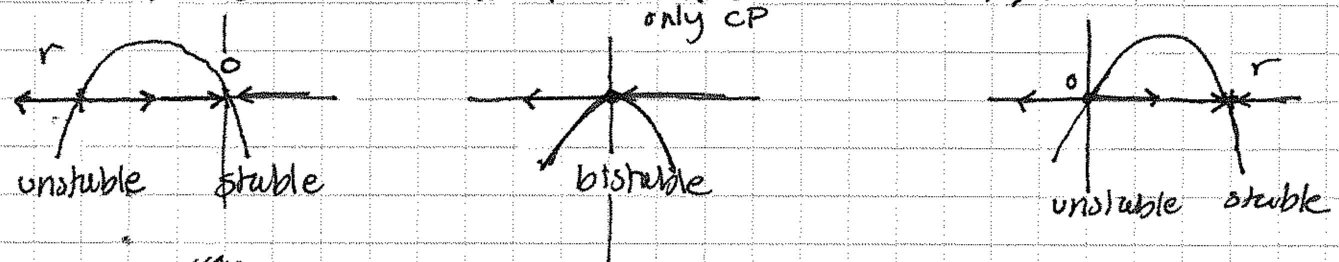
- Depending on the value of a parameter the fixed point changes stability.

EX $\dot{x} = rx - x^2$ This is the normal form for transcritical bifurcations

similar to logistic eqn w/ no harvesting → "growth" parameter.

Fixed points: $rx - x^2 = 0$ $x(r-x) = 0$ $x = 0$ and $x = r$

if $r < 0$ if $r = 0$ $x=0$ only CP $r > 0$



The two fixed points exchanged stabilities!

Plox x vs r for $rx - x^2 = 0$ always have two lines here $x(r) = r$ $x(r) = 0$