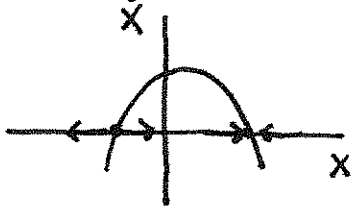
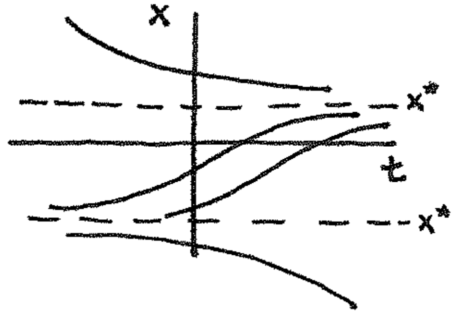
Last time introduced bifurcations:

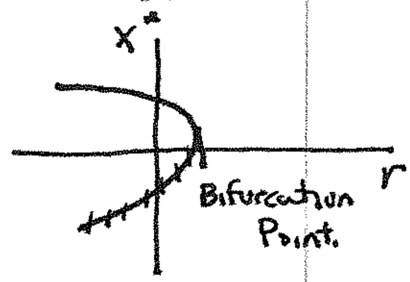
FLOW DIAGRAM



PHASE PORTRAIT



BIFURCATION DIAGRAM



A harder Example:

$$\dot{x} = r - x - e^{-x}$$

Find Fixed Points: $r - x - e^{-x} = 0$

or

$r - x = e^{-x}$ split into two functions that we can plot.

line slope −1 intercept @ r

decaying exponential $x=0$ $e^{-x}=1$

Foxplot:

Two Fixed points · ONE Fixed point · NO Fixed points.

Looks like saddle node.

Bifurcation Point $r=1$.

Meet tangentially at bifurcation point.

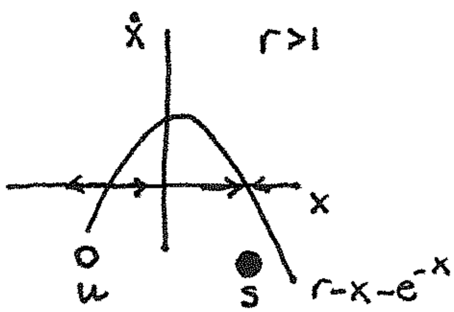
$\frac{d}{dx} r - x = \frac{d}{dx} e^{-x}$

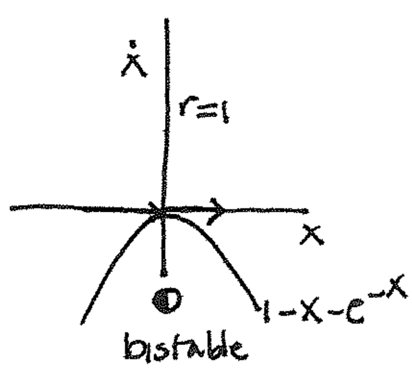
$-1 = -e^{x} \Rightarrow x = 0$ or $r - 0 = 1$

$r=1$ is the point.

Test Stability of Fixed Points:

Flow on Line





$f'(x) = -1 + e^{-x}$

$f'(x^*) = -1 + e^{-x^*}$

problem... don't explicitly know x^*

BUT...

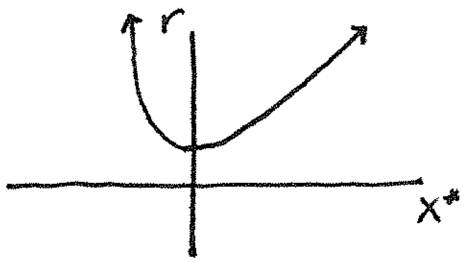
when $x^* < 1$ $e^{-x^*} > 1$ and $-1 + e^{-x^*} > 1$ unstable

when $x^* > 1$ $e^{-x^*} < 1$ and $-1 + e^{-x^*} < 1$ stable

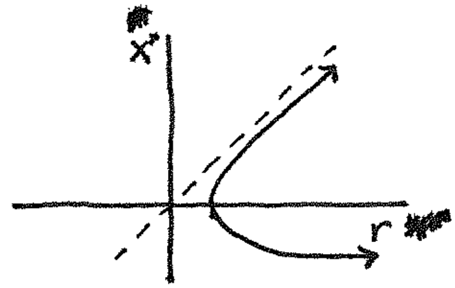
Can't solve explicitly for x as a function of r!

Trick ... we can actually still plot this.

Plot $r = x^* + e^{-x^*}$



Reflect across $r = x^*$



bifurcation point $r = 1$.

Saddle node — fixed points are created or destroyed.

Normal Forms: $\dot{x} = r + x^2$ or $\dot{x} = r - x^2$

<u>Transcritical Bifurcation</u> — has at least one fixed point for all parameter values. The fixed points change stability.

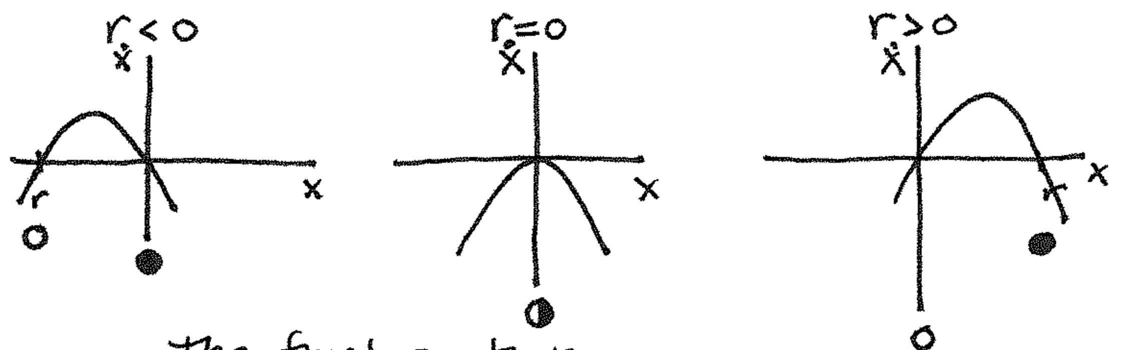
normal form: $\dot{x} = rx - x^2$

Analyze this ... You try on your own ... get ahead.

Fixed Points: $rx^* - x^{*2} = 0$ $\quad x^* = 0$ and $x^* = r$

cases $r < 0$, $r = 0$, $r > 0$

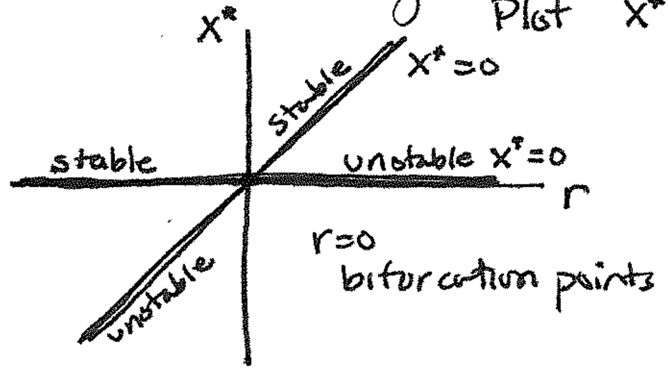
Stability of Fixed Points:

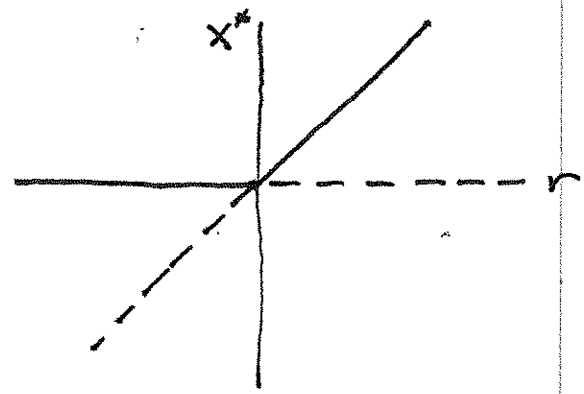


the fixed points move together and exchange stability.

Bifurcation Diagram.

Plot $x^*=0$ and $x^*=r$





Pitchfork Bifurcation — fixed points appear and disappear is symetric pairs. Common in problems with symetry.

Normal Form

1. Supercritical $\dot{x} = rx - x^3$
2. Subcritical $\dot{x} = rx + x^3$

Supercritical Pitchfork

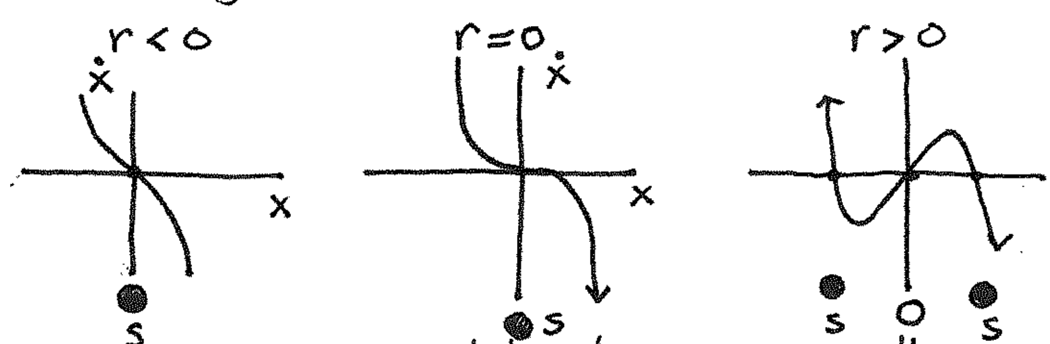
$$\dot{x} = rx - x^3$$

Fixed Points: $rx^* - x^{*3} = 0$

$x^* = 0$ and $x^* = \pm\sqrt{r}$

cases $r<0$ $r=0$ $r>0$

Stability of Fixed Points



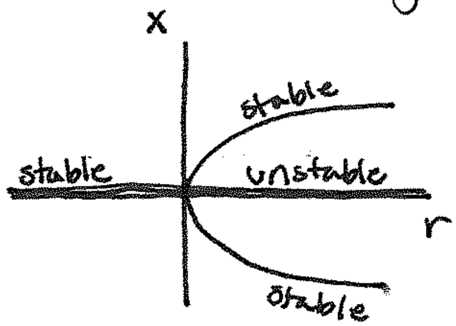
Just barely stable !!

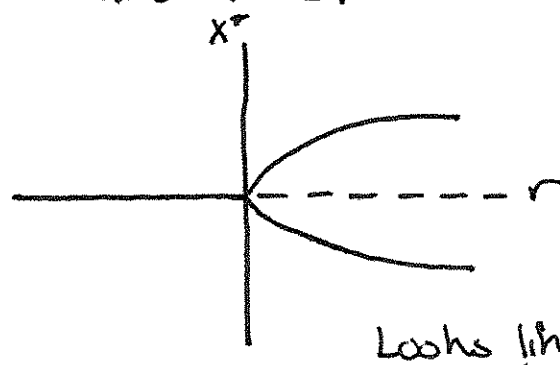
"Critical slowing down"

it is getting very hard to reach $x=0$.

A building about to fall down will wobble taking a long time to come to rest.

Bifurcation Diagram.
Plot $x^*=0$ and $x^*=\pm\sqrt{r}$





r=0 bifurcation point.

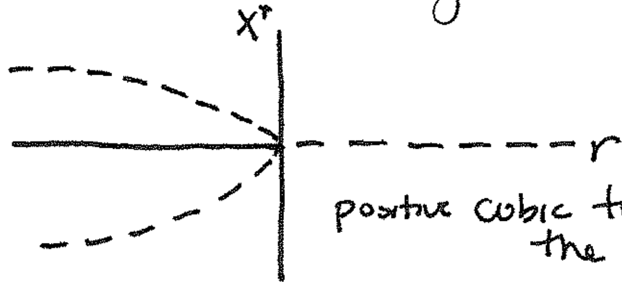
Looks like a pitchfork.

Note the cubic term $-x^3$ was stabilizing.

<u>Subcritical Pitchfork</u> $\dot{x} = rx + x^3$

now the cubic term is destabilizing.

Bifurcation Diagram.



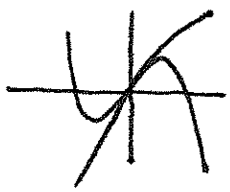
positive cubic term helps to drive the system to ∞.

A much harder example... $\dot{x} = x(1-x^2) - a(1-e^{-bx})$

This does not look like <u>any</u> of our normal forms.
Analyze the system and classify the type of bifurcation.

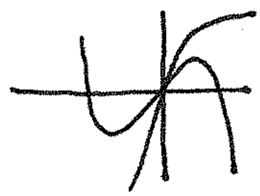
Fixed points — graph on fooplot.
$x(1-x^2) = a(1-e^{-bx})$

if $a=b=1$



$x=0$ fixed point.

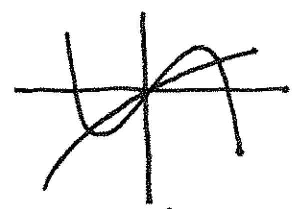
$a=1 \quad b>1$



$x=0$ fixed point.

Both a and b change stability and FP.

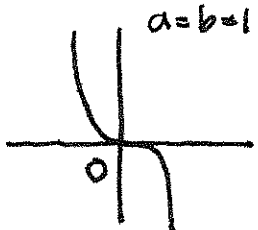
$a=1 \quad b<1$



three fixed points. $x=0$ still a fixed point.

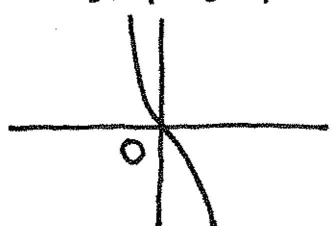
Stability of Fixed Points
Flow on the line

$a=b=1$



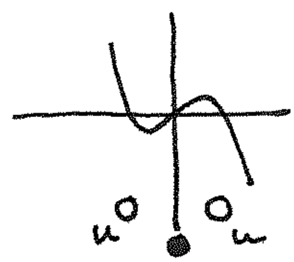
unstable

$a=1 \quad b>1$



unstable

$a=1 \quad b<1$



Something is happening to our fixed point at $x=0$... but what??

Taylor series! We can expand $1-e^{-bx}$ about $x=0$ and look at the leading order terms.

$$\underset{\text{near } x=0}{1-e^{-bx}} = f(0) + f'(0)(x-0) + \tfrac{1}{2}f''(0)(x-0)^2 + \cdots$$

$$= 1-1+bx - \tfrac{1}{2}b^2x^2 + O(x^3)$$

so
$$\dot{x} = x(1-x^2) - abx - \tfrac{1}{2}ab^2x^2 + \cdots$$

$$= x - abx - \tfrac{1}{2}ab^2x^2 + O(x^3)$$

note the $-x^3$ is in here.

$$= (1-ab)x - \tfrac{1}{2}ab^2x^2 + O(x^3)$$

$f(x) = 1 - e^{-bx}$ $\sim$ $0 + bx \mp \frac{1}{2}b^2x^2 + \frac{1}{6}b^3x^3 + \dots$

$f'(x) = be^{-bx}$

$f''(x) = -b^2e^{-bx}$

$\vdots$

Plug in $\dot{x} = x(1-x^2) - a\left[bx - \frac{1}{2}b^2x^2 + \frac{1}{6}b^3x^3 + O(x^4)\right]$

$= (1-ab)x + \frac{1}{2}ab^2x^2 + (1/6\,b^3 - 1)x^3 + O(x^4)$

compare to normal forms:

$\dot{x} = rx - x^2$ $\qquad\qquad$ $\dot{x} = rx - x^3$

It is kinda between a transcritical and a pitchfork. Either way $r=0$ is a bifurcation point.

$(1-ab) = 0 \Rightarrow ab = 1$ is the bifurcation point.

NOTE when $ab = 1$ $\quad \dot{x} = (0)x + \frac{1}{2}x^2 - \frac{5}{6}x^3 + O(x^4)$

↑ the cubic term dominates.

Recall example from the beginning: $\dot{x} = r - x - e^{-x}$

Expand e^{-x} about $x=0$ The FP $\qquad e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + O(x^4)$

so $\dot{x} = r - x - 1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)$

$= (r-1) - \frac{1}{2}x^2 + O(x^3)$

↑ since $\frac{1}{2} > \frac{1}{6}$ put x^3 term in here

this matches saddle node Normal form $\quad \dot{x} = r - x^2$

so when $(r-1) = 0$ or $r=1$ we expect a bifurcation.