Last time: Bifurcations.

Saddle Node
Transcritical
Pitchfork

1. Find the fixed points and decide on cases for the parameter
2. Analyze the stability of the fixed points
   a. Flow on a Line
   or b. Linear stability.
3. Draw Bifurcation Diagram using F.P. equations from Part 1.
4. Describe your results – relate back to the "real" world

TODAY – more examples and interesting situations.

<u>Ex</u> $\dot{x} = rx + x^3 - x^5$

recall: last time x^3 was destabilizing! subtracted higher order terms, $-x^5$, help to stabilize.

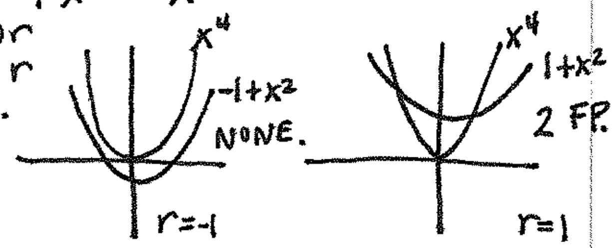
1. Fixed Points: $rx + x^3 - x^5 = 0$ $\quad x^* = 0$ is always a fixed point.

$x(r + x^2 - x^4) = 0$

[Expect up to 5 F.P.] $\quad r + x^2 - x^4 = 0$

$r + x^2 = x^4$

Graph for different r values.



CASES: $r < -\frac{1}{4}$

$-\frac{1}{4} < r < 0$

$r > 0$

look for where the two graphs are Tangent...

$\frac{d}{dx}\, r + x^2 = 2x = \frac{d}{dx} x^4 = 4x^3$

TWO CASES: $\frac{2}{4} = x^2 \qquad x = \pm\frac{1}{\sqrt{2}}$

Then $r + \left(\pm\frac{1}{\sqrt{2}}\right)^2 = \left(\pm\frac{1}{\sqrt{2}}\right)^4$

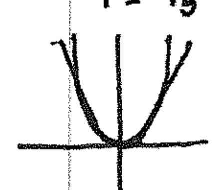
$r + \frac{1}{2} = \frac{1}{4} \qquad r = -\frac{1}{4}$

Bifurcation Point $\boxed{r = -\frac{1}{4}}$

<u>OR</u> $2x = 4x^3$ when $x = 0$

$r + (0)^2 = (0)^4 \quad \boxed{r = 0}$ Bifurcation Point

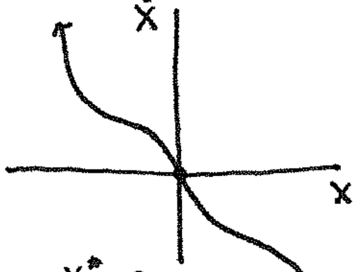
$r = -1/5$



actually 4 FPs!

2. Analyze Stability... Flow on the line

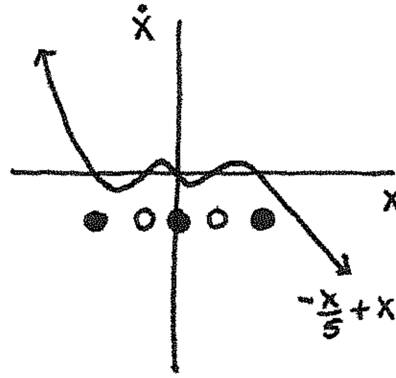
$r < -1/4$
let $r = -1$



$x^* = 0$
Stable F.P.

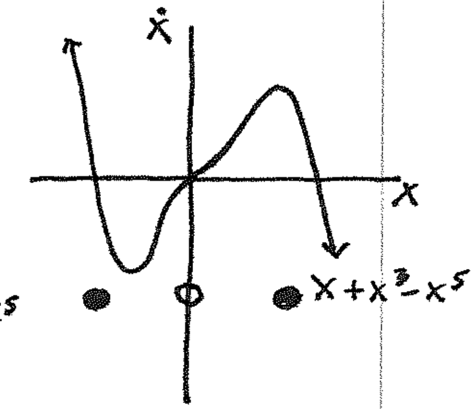
$-x + x^3 - x^5$

$-1/4 < r < 0$
let $r = -1/5$



$x^* = 0$ Stable
5 fixed points

$r > 0$
let $r = 1$

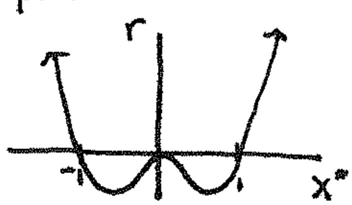


$x^* = 0$ unstable
3 fixed points

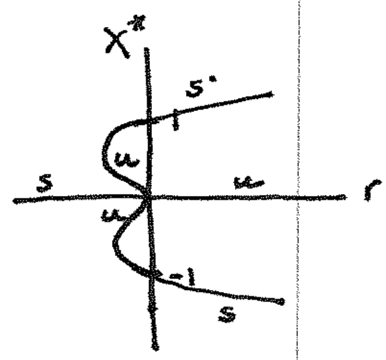
3. Bifurcation Diagram.

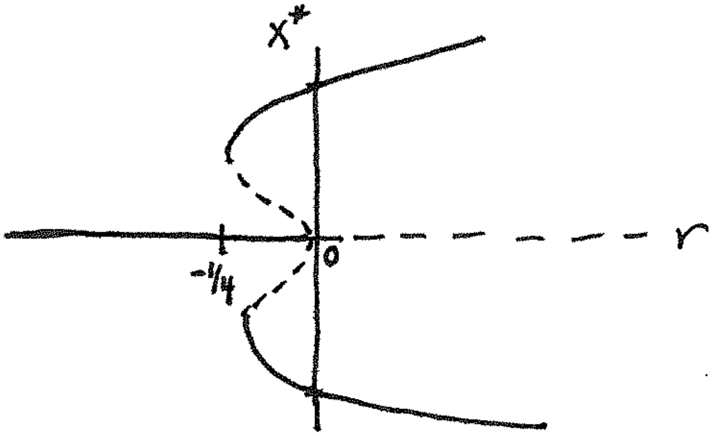
Draw $x^* = 0$ and $r + x^2 - x^4 = 0$

USE TRICK
plot $r = x^4 - x^2$



Flip x^* and r





Bifurcation Points at
$r = -1/4$ and $r = 0$

The $-x^5$ helps keep solutions from $\to \pm\infty$
Stabilizing!

4. Comments — This Diagram has some interesting Features.

Physical System... Start with $r < -1/4$

Then ~~the~~ the soln goes toward $x^* = 0$

Now we slowly start turning our parameter dial up.

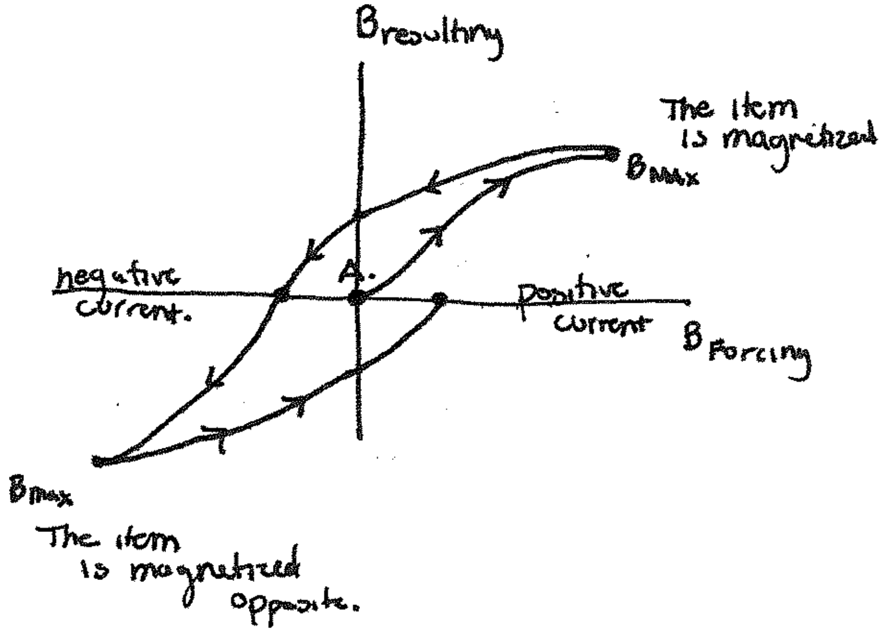
$r \to 0^-$ then what happens?

- The soln jumps to an upper or lower branch.

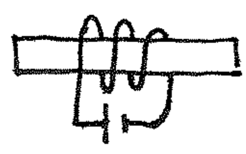
We then turn the dial back $r \to 0^+$ but we have to keep going! $r \to -1/4$ before we get back to $x^* = 0$.

This is called <u>HYSTERESIS</u>

Video — magnetic Hysteresis.



A. Start w/ unmagnetized item and put it in a magnetic field.



Current in Loop Creates Magnetic Field.

It becomes magnetized and remains so even after the current stops

* We also see this in Economic or Natural Systems!!
Relationship between Inflation and Unemployment.

Group Work Day 7

Consider the system:

$$\dot{x} = rx + \frac{x^3}{1+x^2}$$

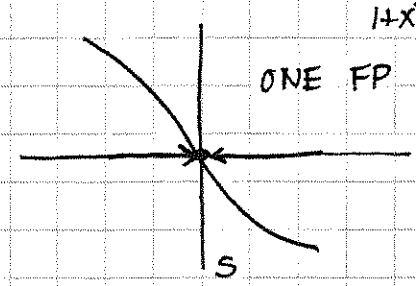
Your goal is to completely analyze the system. Please do the following:

1. Solve for the fixed points. Be prepared to discuss how you found your fixed points.
2. Decide on the possible cases for r. How did you figure this out?
3. Draw FLOW diagrams for each distinct case for r.
4. Say in words what is happening to the fixed points. Stable? Unstable? Moving toward or away?
5. Draw the bifurcation diagram (x^* vs. r), remember to include stability information. Be prepared to explain how you got the graph that you did.
6. From looking at the Bifurcation Diagram, say what is happening in the system for a variety of initial x values and choices for r.

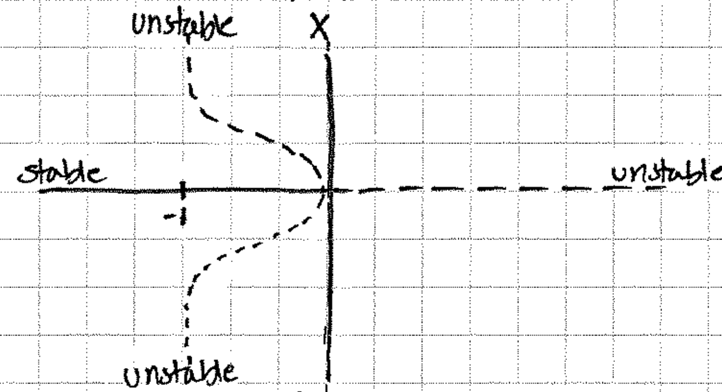
When we get back together to present to the class I will assign each group to present one part of the problem. You are still expected to do all parts of the problem.

$r < -1$ let $r = -2$

plot $-2x + \frac{x^3}{1+x^2}$



#5 NOW REDRAW BIFURCATION DIAGRAM...



This would be a subcritical pitchfork bifurcation

$r=0$ bifurcation point

$r=-1$ also bifurcation point.

EXTRA EXAMPLE...

<u>Ex</u> $\dot{x} = x + \tanh(rx)$ $\qquad$ $\tanh(x)$ = hyperbolic tangent.

<u>ASIDE</u>: Hyperbolic Functions:

Alternate form for Sin / Cos $\qquad$ $\sin(x) = \frac{e^{+ix} - e^{-ix}}{2i}$ $\qquad$ $\cos(x) = \frac{e^{+ix} + e^{-ix}}{2}$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \qquad \cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

then $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

Just like $\sin(x)$ and $\cos(x)$ we can do inverses of the hyperbolic functions.

#1 FIXED POINTS:

$x + \tanh(rx) = 0$ $\qquad$ $\tanh(x) = 0$ when $x = 0$

so $x=0$ is always an FP.

solve for r: $\quad r = \frac{\operatorname{arctanh}(-x)}{x}$

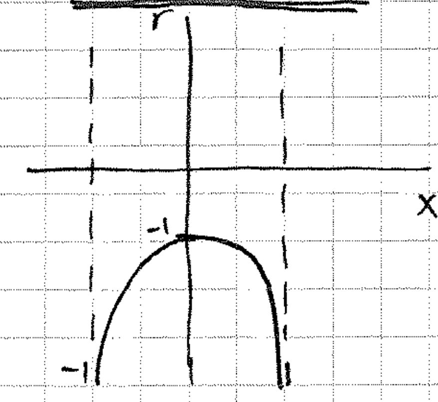
#2 TRY TO UNDERSTAND OTHER FP's

can't really solve much here!

On Wolfram: $\operatorname{arctanh}(-x)$ real for $-1 < x < 1$

#3 PLOT x vs r : This gives bifurcation diagram with out stability.

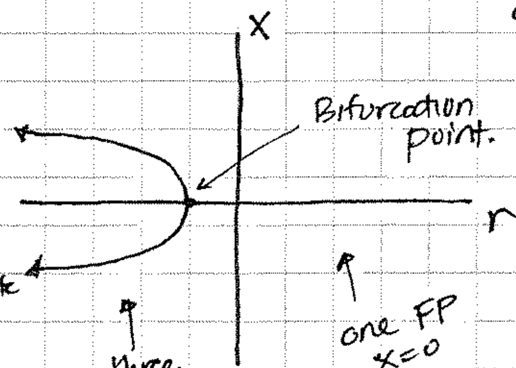
First r vs x



THEN FLIP TO GET →

when x=1
r = large asymptote
x=0 r=-1
x=-1 r= large asymptote

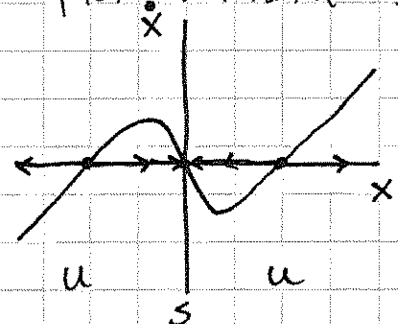
x vs r — the bifurcation diagram



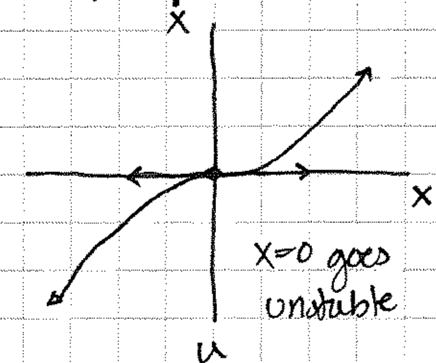
#4 NOW TEST STABILITY... CASES $r<-1$, $r=-1$, $r>-1$

FLOW ON THE LINE.

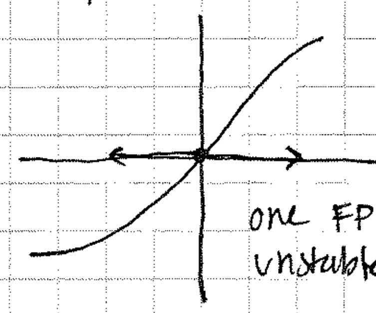
$r<-1$ expect three FP
choose $r=-2$
plot $x+\tanh(-2x)$



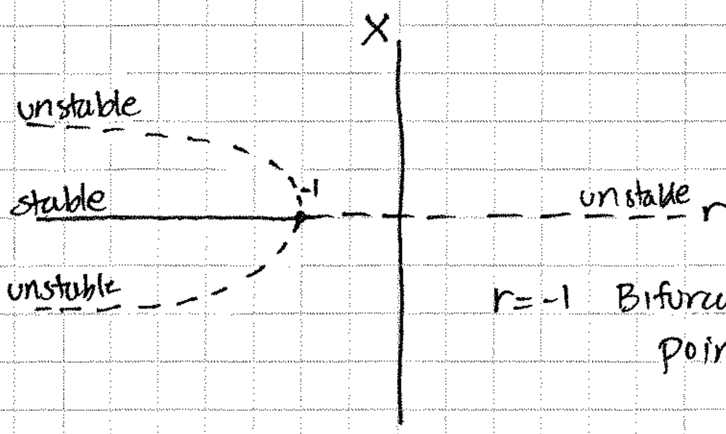
$r=-1$ expect Bifurcation
plot $x+\tanh(-x)$



$r>-1$ expect one FP
choose $r=+1$
plot $x+\tanh(x)$



#5 DRAW BIFURCATION DIAGRAM w/ STABILITY.



r=-1 Bifurcation point

This is a subcritical pitchfork bifurcation.