Imperfect Bifurcation and Catastrophy.

Introduce a New Graph!

Ex $\dot{x} = h + \underbrace{rx - x^3}$

we add an extra parameter
This bereaks the symmetry.

this looks like our symmetric pitchfork normal form.

$h \neq 0$ then h is an imperfect parameter.

We will analyze this system for both parameters
LOTS OF CASES!!

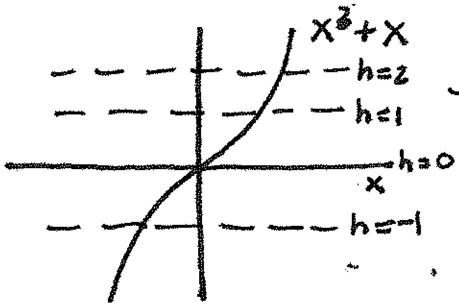
1. Solve for Fixed Points!

$h + rx - x^3 = 0$ separate:

$$h = x^3 - rx$$

Cases for r: $r<0$ $r=0$ $r>0$

$r<0$
let $r=-1$



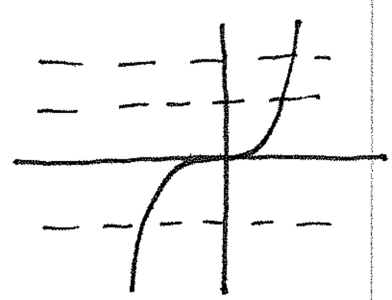
look for intersections of h and $x^3 - rx$

then $h =$ constant just a bunch of horizontal lines.

$r<0$ always 1 FP.
as h decreases the value of the FP decreases

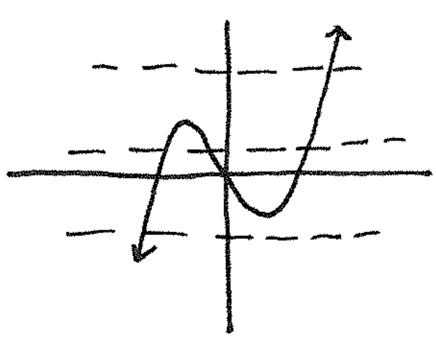
$r=0$

Same as $r<0$



one F.P.

$r>0$
let $r=1$



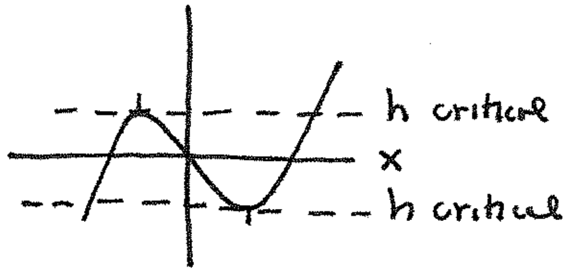
More complicated...
For some values of h One FP... but some values of h have Three F.P.

Summarize F.P.

EASY — when $r \le 0$ One F.P. For all values of h.

HARD — when $r > 0$ more complicated... Bifurcations happen when the # of F.P. changes.

How can we solve for the "critical" h values when $r > 0$



These happen when $x^3 - rx$ reaches a local max or min.

From Calculus! Find max and min. of $x^3 - rx$

Take derivative — set to zero

$$\frac{d}{dx}\left[x^3 - rx\right] = 3x^2 - r = 0$$

$$r = 3x^2 \quad \text{or when} \quad x = \pm\sqrt{\frac{r}{3}}$$

Plug into $h = -rx + x^3$ to find critical h values.

$$h = r\left[\pm\sqrt{\frac{r}{3}}\right] - \left[\pm\sqrt{\frac{r}{3}}\right]^3$$

$$= \pm \frac{r\sqrt{r}}{\sqrt{3}} \mp \frac{r\sqrt{r}}{3\sqrt{3}} = \frac{\pm 3r\sqrt{r} \mp r\sqrt{r}}{3\sqrt{3}}$$

$$= \pm \frac{r\sqrt{r}\left[3 - 1\right]}{3\sqrt{3}} = \pm \frac{2r\sqrt{r}}{3\sqrt{3}}$$

So Bifurcation points happen at

$$h = \pm \frac{2r}{3}\sqrt{\frac{r}{3}}$$

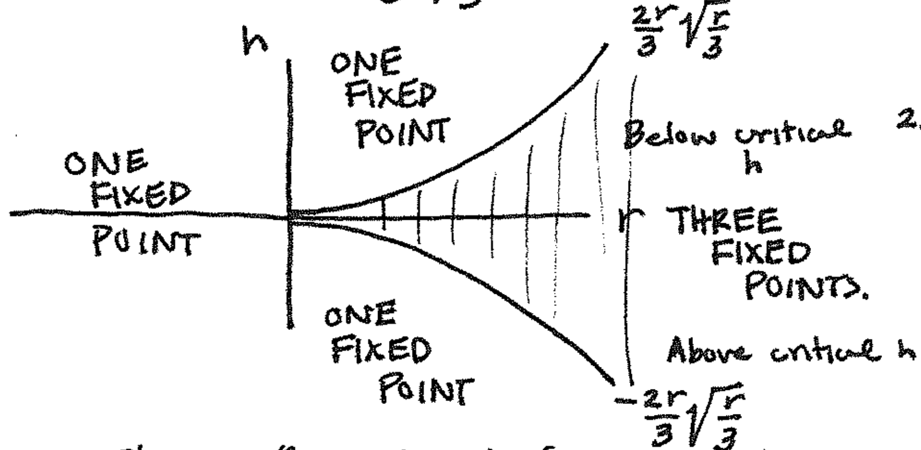
when $r > 0$

really this is a bifurcation CURVE.

★ INTRODUCE <u>STABILITY</u> <u>DIAGRAM</u>!
- analyze two parameters at one time.

Plot h vs r and then write in stability information!

1. draw $h = \pm \frac{2r}{3}\sqrt{\frac{r}{3}}$



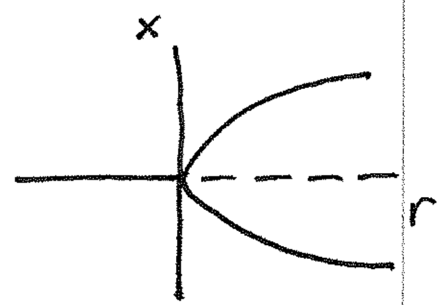
2. Write in the stability info... How many Fixed Points.

Shows the # of fixed points as we move around parameter space!

So we know the # of Fixed Points ... whats next?

2. Stability of Fixed Points.

If $h=0$ Easy $\dot{x} = rx - x^3$
Typical Pitchfork

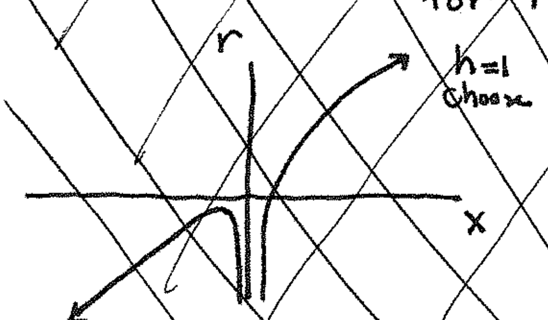


when $h \neq 0$ we have some cases:

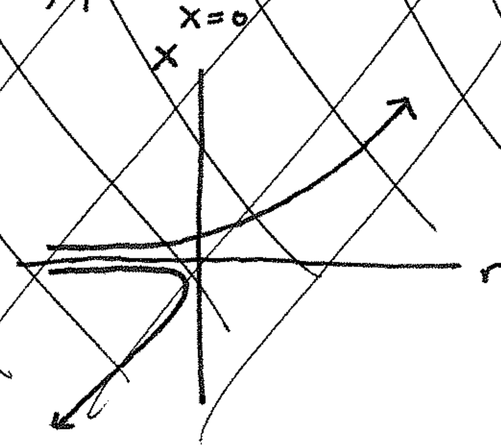
Fixed h: Plot x^* vs r

$h + rx - x^3 = 0$ solve for r $\quad r = \frac{x^3 - h}{x}$ use the flip trick.

asymptote at $x=0$



FLIP

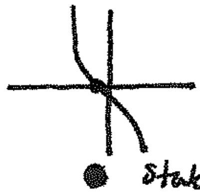


When $h \neq 0$ we have some cases!

CASE $r < 0$ let $r = -1$
expect one FP for all h.

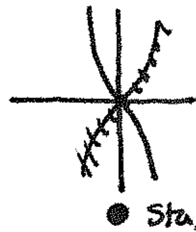
FLOW ON LINE

$h < 0$



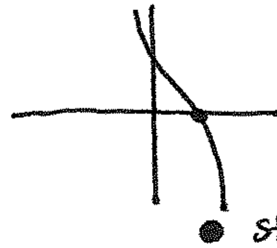
stable

$h = 0$



Stable

$h > 0$



stable.

SO when $r < 0$ we have one Fixed Point and it is stable.

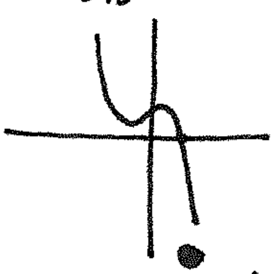
CASE $r = 0$ still expect one FP for all h.
FLOW ON LINE
again we would find always stable.

SO when $r = 0$ we have one stable Fixed Point.

CASE $r > 0$ Expect one or three FP. let $r = 1$

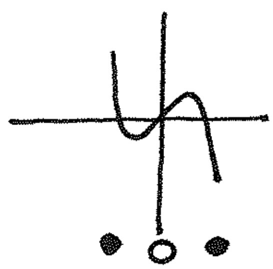
then $h = \pm \frac{2}{3\sqrt{3}} \sim \pm .3849.$ is the critical value.

$h > \frac{2}{3\sqrt{3}}$ let $h = .5$



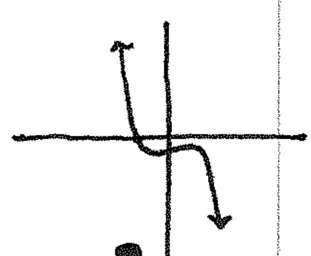
one stable FP.

$\frac{2}{3\sqrt{3}} > h > \frac{-2}{3\sqrt{3}}$ let $h = .1$



three FP

$h < \frac{-2}{3\sqrt{3}}$ let $h = -.5$



one stable.

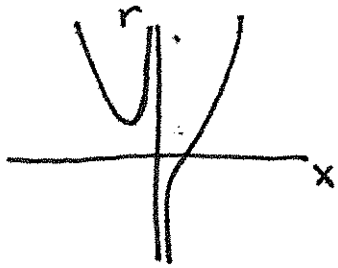
SO outside of critical range one stable FP
inside Three FP The middle one is unstable.

Put this all together in Bifurcation Diagrams for Positive vs. Negative h.

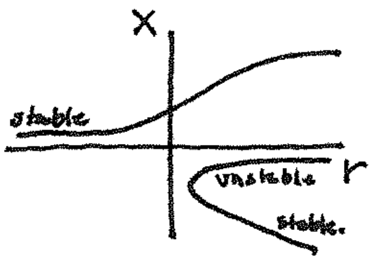
plot $r = \frac{x^3 - h}{x}$ then FlIP.

$h > 0$

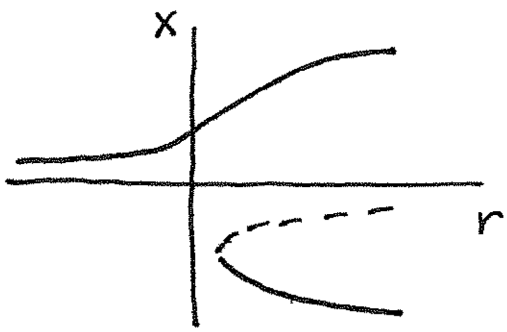
let $h = 1$



FlIP

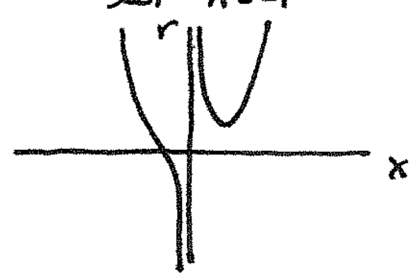


ADD STABILITY

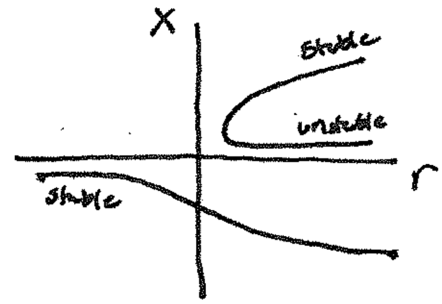


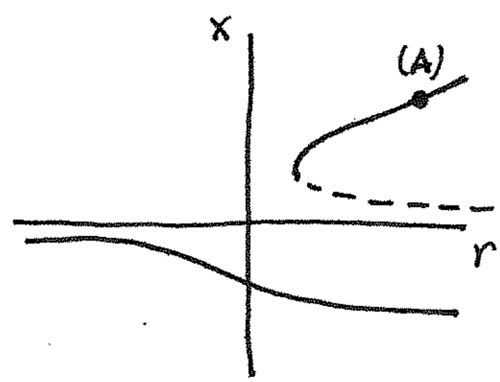
$h < 0$

let $h = -1$



FLIP





In real Physical systems you can usually say $h > 0$ or $h < 0$ only — because one does not make physical sense.

CATASTROPHY.

- Instead of a smooth transition Imagine you start at (A) then slowly turn down your parameter r.
- At some point you drop off the edge to the lower solution curve and even increasing r at that point will not get you back to the top solution curve.

$$\dot{N} = RN\left(1 - \frac{N}{k}\right) - \frac{BN^2}{A^2+N^2}$$

too many parameters to deal with individually!

NON-DIMENSIONALIZATION – a trick of applied math!

Goal – define new variables that simplify the equation.

Non-unique – the is more than one correct choice for the variables... in real life we try lots of ways and choose the best one.

1. Define the new variables...

let $x = \frac{N}{A}$ be our new dependent variable.

and $\tau = \frac{B}{A}t$ be our new independent variable.

2. Sub these new variables in one at a time.

If $x = \frac{N}{A}$ then $N = Ax$ and $\dot{N} = A\dot{x}$

our equation becomes $A\dot{x} = RAx\left(1 - \frac{Ax}{k}\right) - \underbrace{\frac{BA^2x^2}{A^2+A^2x^2}}_{\text{good sign – things cancel.}}$

$$A\dot{x} = RAx\left(1 - \frac{Ax}{k}\right) - \frac{Bx^2}{1+x^2}$$

then $\tau = \frac{B}{A}t$ and $t = \frac{A}{B}\tau$

need to substitute for $\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau}$

notice the $d\tau$ "cancel"

so $\frac{d}{dt} = \frac{B}{A}\frac{d}{d\tau}$ I can sub for $\dot{x} = \frac{dx}{dt}$

$$A\frac{B}{A}\frac{dx}{d\tau} = RAx\left(1 - \frac{Ax}{k}\right) - \frac{Bx^2}{1+x^2}$$

simplify this...

$$\frac{dx}{d\tau} = \frac{RA}{B}x\left(1-\frac{A}{K}x\right) - \frac{x^2}{1+x^2}$$

3. Define new "simpler" parameters:

$$r = \frac{RA}{B} \quad \text{and} \quad k = \frac{K}{A}$$

$$\boxed{\dot{x} = rx\left(1-\frac{x}{k}\right) - \frac{x^2}{1+x^2}}$$

analyze this equation in terms of r and k.