

Today we start two dimensional flows.

* We will need some ideas from Linear Algebra.
- matrix multiplication
- eigenvalues.

* I am going to skip some of the more complicated ideas of solving these systems using eigenvalues and vectors

We first need to understand LINEAR SYSTEMS.

Consider the 2-D system

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}$$

this is a linear system

Why?

How would we make it nonlinear.

parameters a, b, c, d variables $x(t)$ and $y(t)$.

Recall matrix multiplication:

$$\begin{bmatrix} \rightarrow \\ \rightarrow \end{bmatrix} \begin{bmatrix} \downarrow \\ \downarrow \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (1 \times 2) + (2 \times 3) \\ (3 \times 2) + (4 \times 3) \end{bmatrix} = \begin{bmatrix} 2 + 6 \\ 6 + 12 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$$

We can write our system in matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} (a \times x) + (b \times y) \\ (c \times x) + (d \times y) \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

matrix mult should give me back the system.

Sometimes we write:

$$\dot{\vec{x}} = A \vec{x} \quad \text{where } \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

NOTE: be careful of the order:

$$\left| \begin{array}{l} \dot{y} = cx + by \\ \dot{x} = ax + by \end{array} \right. \text{ rewrite } \begin{array}{l} \dot{y} = by + cx \\ \dot{x} = by + ax \end{array}$$

$$\left| \text{then } \begin{bmatrix} \dot{y} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} \right.$$

YOU TRY:

write the mass on a spring equation in matrix form:

$$m\ddot{x} + kx = 0$$

linear spring no friction!

- let them get ahead then do calc...
need x_1 and x_2

let

$$\dot{x}_1 = x_2$$

$$\text{then } \dot{x}_2 = \ddot{x}_1$$

solve original eqn for $\ddot{x}$

$$\ddot{x} = -\frac{kx}{m}$$

$$\text{let } x = x_1$$

$$\ddot{x}_1 = -\frac{kx_1}{m}$$

so

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1$$

in matrix form:

NOTE: $\dot{x}_1 = 0x_1 + x_2$
 $\dot{x}_2 = -\frac{k}{m}x_1 + 0x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

you can always
double check
this by doing
matrix multiplication.

PHASE PLANE — in two dimensions our solution is
a flow in the plane.

[as opposed to flow on the line]

1. Find fixed points:

Need BOTH $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ AT SAME TIME.

$$\dot{x}_1 = x_2 = 0 \quad \dot{x}_2 = -\frac{k}{m}x_1 = 0$$

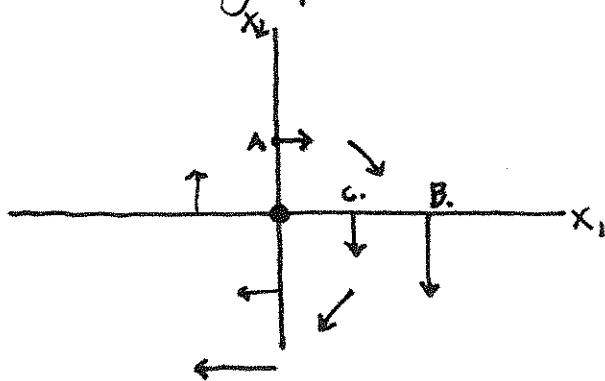
Fixed point $(x_1, x_2) = (0, 0)$

The origin on the plane

2. Draw the phase plane...

Flow in the plane defined by the vector
 $(\dot{x}, \dot{y}) \rightarrow (x_2, -\frac{k}{m}x_1)$

draw a graph of x_2 vs x_1



$(0,0)$ is my fixed point

- choose other points to examine the flow:

A. $(0,1)$ $x_1=0$ $x_2=1$

plug into

$$\langle x_2, -\frac{k}{m}x_1 \rangle \rightarrow \langle 1, 0 \rangle$$

vector points only toward positive x_1

Keep doing this for a lot of points in the plane.

B. $(2,0)$ so the flow is $\langle 0, -\frac{k}{m}2 \rangle$

C. $(1,0)$ $\langle 0, -\frac{k}{m}1 \rangle$

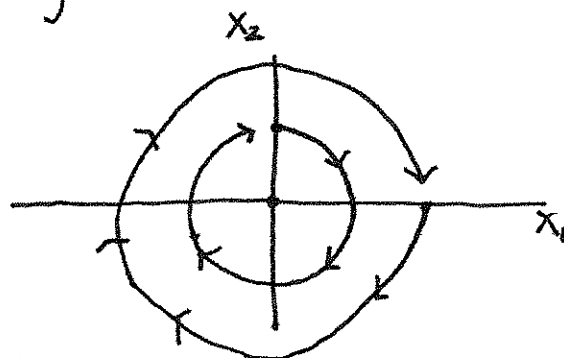
The flow gets "stronger" the further out we are.

The vectors show both the speed and the direction of the flow.

Next... start with an initial condition $(x_1(0), x_2(0))$ and think about what the flow would do to a particle at that point.

$(x_1(0), x_2(0)) = (0,0)$ - at the fixed point nothing happens.

drawing the PHASE PLANE again.



starting at A.
or at B.

This makes Physical Sense

• The spring oscillates!

$\rightarrow \square \leftarrow \square \leftarrow \square$ etc.

what do x_2 and x_1 represent physically:

$x_1 = x$ position

$x_2 = \dot{x}_1 = \dot{x}$ velocity

Think through one cycle...

B. Start w/ zero velocity positive position... what happens!

Useful tool — ON MATLAB ... PPLANE.
Demo...

let $x_1 = x$ and $x_2 = y$
define k and m as parameters.

set symmetric window.

GROUP WORK:

Investigate on Pplane $\dot{x}_1 = ax_1$
 $\dot{x}_2 = -x_2$

we call these eqns "uncoupled"

How would we couple ^{why?} them?

Analysis of an Uncoupled Linear System

Together we just saw a linear system that had closed circular orbits in the phase plane. Now we will investigate a different system to see what other possible curves we might see in the phase plane.

Consider the system:

$$\dot{x}_1 = ax_1, \quad \dot{x}_2 = -x_2$$

where a is a parameter that can take on any real values. What is the fixed point?

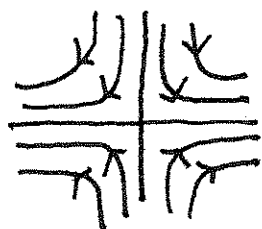
Your job is to plot the phase plane for a variety of a values as see if you can come up with ALL of the qualitatively different solutions possible for this system and ranges for a . For each new type of phase plane do the following:

- Draw the phase plane in your notes or on this paper. (copy what you see in the Matlab graph) Make sure that you include arrows to show the direction of the solution.
- Say in words what is happening to your solution. EX. When I start at $(1, 0)$ my solution goes directly to the fixed point along the x_2 axis with no change in x_1 , but if I start at $(1, 1)$ my solution decays to the fixed point decaying faster in the x_1 direction than in the x_2 direction.
- Solve the system of equations (you can solve each separable equation independently).
- Do your pictures make sense base on the solution you just found? (Hint: talk about the parameter a and compare the exponents.)

When we get back together as a class we will develop vocabulary to identify each of these different looking graphs. So leave room on your paper to add descriptions during lecture.

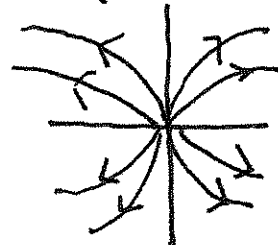
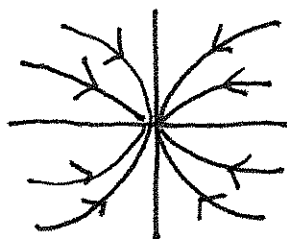
Possible Types of Fixed points in the linear system:

Saddle points
(unstable)



one exponent
pos the other neg

Nodes (Proper)
(stable) (unstable)



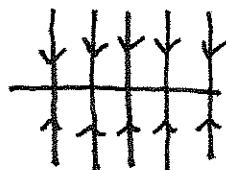
exponents have same sign.

Stars or Degenerate Nodes



SPECIAL
CASES

Line of Fixed points.



when one
exponent
is zero.

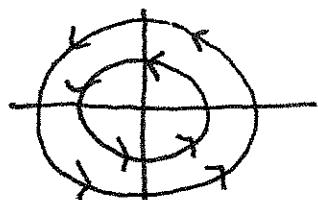
∞ # of fixed points



They happen
when your
exponents are
equal...

stability depends on sign of exponent.

centers

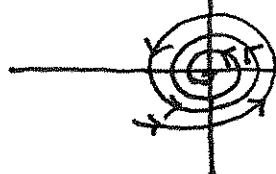


exponents are
imaginary #'s

oscillations

"spring
mass"

Spirals
(stable)



Exponents are
complex #'s

(unstable)



So the dynamics of the system depends on... THE EXPONENT
in $e^{\lambda t}$

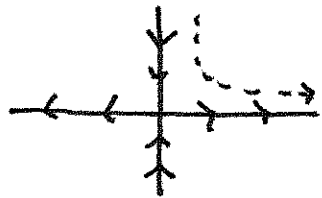
λ - eigenvalue.

Next time we learn to solve for these.

Some important vocabulary...

Manifolds — They divide phase space... solutions cannot cross manifolds.

For the Saddle Point



unstable manifold — The set of all initial conditions such that $x \rightarrow x^*$ as $t \rightarrow -\infty$

stable manifold — The set of all initial conditions such that $x \rightarrow x^*$ as $t \rightarrow \infty$

In General
Solutions approach the unstable manifold as $t \rightarrow \infty$... draw some flows to confirm this.

Attracting Fixed Point — All trajectories that start near and approach it as $t \rightarrow \infty$

If all trajectories approach it (no matter the initial condition)
Then it is LOCALLY ATTRACTING.

"STABLE"

Liapunov stable — all trajectories starting sufficiently close remain close for all time.

⊕ "NEUTRALLY STABLE" — The point doesn't attract the sys but also doesn't repel it!

IF a system is BOTH Liapunov stable and attracting we call it asymptotically stable.

— as far as all of our fixed points
1-D stable = asymptotically stable.

IF a system is NEITHER Liapunov or attracting we call it unstable