Reduction to a First Order System

$$y'' + 6y' + 9y = 0$$

$$y_1' = y_2$$

$$y_2' = -6y_2 - 9y_1$$

★ $$y^{(4)} - 8y'' + 16y = 0$$

$$y_1' = y_2, \quad y_2' = y_3$$

$$y_3' = y_4, \quad y_4' = 8y_3 - 16y_1$$

$$y''' + 2y' - y = \cos(t)$$

$$y_1' = y_2, \quad y_2' = y_3, \quad y_4' = 1$$

$$y_3' = -2y_2 + y_1 + \cos(y_4)$$

Separable First Order ODE's

$$\dot{y} + 2y = 0$$

$$y = Ae^{-2t}$$

★ $$\dot{y} = \frac{1}{\sin y}$$

$$y = \arccos(c - t)$$

$$\dot{y} = ty^3$$

$$y = \pm\sqrt{\frac{1}{C - t^2}}$$

$$y^3\dot{y} = (y^4 + 1)\cos t$$

$$y^4 + 1 = Ae^{4\sin t}$$

or

$$\tfrac{1}{4}\ln|y^4 + 1| = \sin t + C$$

Flow's on the Line

1. For the following exercise consider the ODE $\dot{x} = \sin(x)$.

- Find all the fixed points of the flow
- At which points x does the flow have the greatest velocity to the right?
- Find the flow's acceleration $\ddot{x}$ as a function of x.
- Find the points where the flow has maximum positive acceleration.

2. Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch typical solution curves for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's is impossible to solve the equation in closed form!

$$\dot{x} = 4x^2 - 16$$

$$\dot{x} = 1 - x^{14}$$

$$\dot{x} = x - x^3$$

HOMEWORK SOLNS

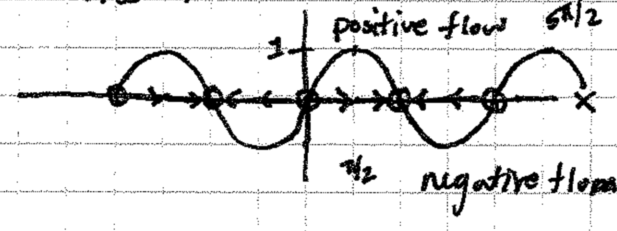
1. Find all the fixed points of the flow

$$\dot{x} = \sin(x)$$

Fixed points $\Rightarrow$ $\dot{x} = 0$ or $\sin(x) = 0$

This happens when $x = n\pi$
n = integer

Flow on the line



2. At what points does the flow have the greatest velocity to the right.

where is the greatest positive flow...
where $\sin(x)$ = maximum

when $x = \frac{(4n+1)\pi}{2}$ n = integer.

3. Find the flows acceleration $\ddot{x}$

chain Rule

take the derivative $\frac{d}{dt}\dot{x} = \frac{d}{dt}\sin(x) = \cos(x)\cdot\dot{x}$

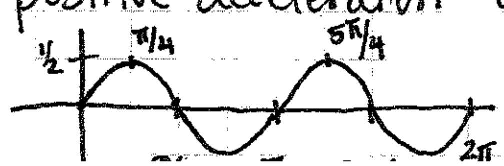
$= \cos(x)\sin(x)$
plug in for $\dot{x}$

$\ddot{x} = \cos(x)\sin(x)$

but $\sin(2x) = 2\cos(x)\sin(x)$ trig identity

$\ddot{x} = \frac{1}{2}\sin(2x)$

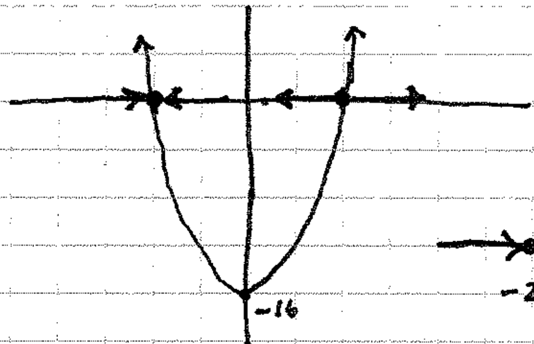
4. Maximum positive acceleration when $\frac{1}{2}\sin(2x)$ = max

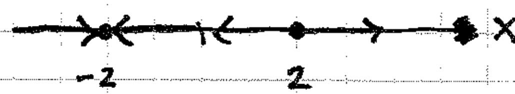


when $x = \frac{(4n+1)\pi}{4}$

5(a) $\dot{x} = 4x^2 - 16$ fixed points $\dot{x} = 0$

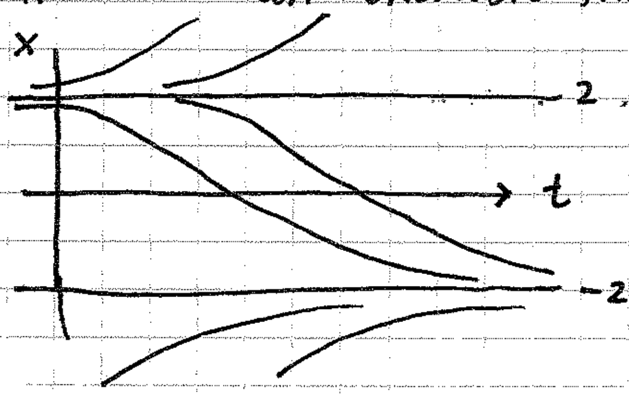
$$4x^2 - 16 = 0$$
$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$





flow on a line.

$x = -2$ is a stable fixed point
$x = 2$ is an unstable fixed point



Solution curves tend away from $x=2$ and toward $x=-2$.

separate terms.

$$\frac{dx}{dt} = 4x^2 - 16 \quad \text{seperable} \rightarrow \frac{dx}{4x^2-16} = dt$$

integrate both sides. $\displaystyle\int \frac{1}{4x^2-16}\,dx = \int dt = t + C$

Partial Fractions.

$$\frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$1 = A(x-2) + B(x+2)$$
$$= (A+B)x + 2(B-A)$$

$A+B=0$ $\quad$ $2(B-A)=1$
$A=-B$ $\quad$ $B-A = 1/2$
$2B = 1/2$
$A = -1/4$ $\quad$ $B = 1/4$

Factor denom. $\displaystyle\frac{1}{4x^2-16} = \frac{1}{4(x^2-4)} = \frac{1}{4(x+2)(x-2)}$

mult by 4 $\displaystyle\int \frac{1}{(x+2)(x-2)}\,dx = 4t + \tilde{C}$

need partial fractions.

$$\int \frac{-1}{4(x+2)}\,dx + \int \frac{1}{4(x-2)}\,dx = 4t + \tilde{C}$$

solve

$$\frac{1}{4}\left[-\ln(x+2) + \ln(x-2)\right] = 4t + \tilde{C}$$

log rules

$$\ln\left(\frac{x-2}{x+2}\right) = 16t + C^*$$

raise e

$$\frac{x-2}{x+2} = Ae^{16t}$$

now solve for x...

$x-2 = Ae^{16t}(x+2)$ mult by denom.

$x - Ae^{16t}x = 2Ae^{16t} + 2 = 2(Ae^{16t}+1)$ factor and gather terms

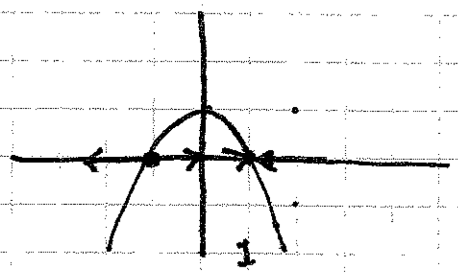
$x(1-Ae^{16t}) = 2(Ae^{16t}+1)$ isolate x and solve.

$$\boxed{x(t) = \frac{2(Ae^{16t}+1)}{1-Ae^{16t}}}$$ closed form soln.

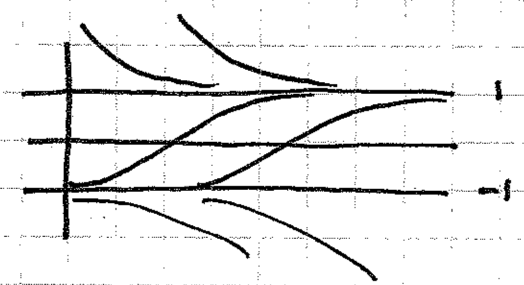
Could plot on Matlab for different solutions.
$x(0)=0$, $x(0)=3$, $x(0)=-3$ etc...

5(b) $\dot{x} = 1 - x^{14}$

$1-x^{14}=0$ when $x^{14}=1$ or $x = 1,-1$



$x=-1$ is an unstable fixed point
$x=1$ is a stable fixed point.



$\frac{dx}{dt} = 1-x^{14}$ seperable so separate $\frac{dx}{1-x^{14}} = dt$

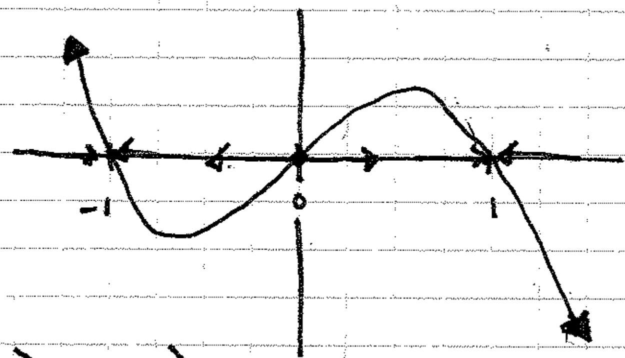
$$\int \frac{dx}{1-x^{14}} = \int dt$$

↳ this is actually really nasty ... plug it into the integrator to find out!

5(c) $\dot{x} = x - x^3$

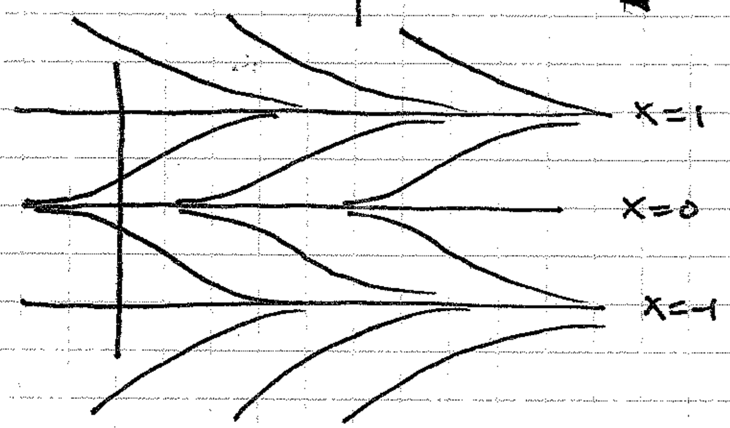
fixed points $\dot{x} = 0 \quad x - x^3 = 0$

$x(1-x^2) = 0 \qquad x(1-x)(1+x) = 0 \qquad$ $x = 0$, $x = 1$, $x = -1$



$x = 0$ unstable fixed point

$x = -1$ and $x = 1$ stable fixed points



Typical solution curves.

$\frac{dx}{dt} = x - x^3$ seperable $\frac{dx}{x - x^3} = dt$

$$\int \frac{1}{x(1-x)(1+x)}\, dx = \int dt$$

↳ this is doable w/ partial fractions...

$$\frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

$$1 = A(1-x)(1-x) + Bx(1+x) + Cx(1-x)$$
$$= A(1-x^2) + B(x+x^2) + C(x-x^2)$$
$$= (B - A - C)x^2 + (B+C)x + (A) = 1$$

$A = 1$
$B + C = 0 \qquad B = -C$
$B - A - C = 0$
$B - C = 1$
$2B = 1 \qquad B = 1/2$
$C = -1/2$

$$\frac{1}{x(1-x)(1+x)} = \frac{1}{x} + \frac{1/2}{(1-x)} - \frac{1/2}{(1+x)}$$

$$\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{1-x}\,dx - \frac{1}{2}\int \frac{1}{1+x}\,dx = t + c$$

$$\ln|x| - \tfrac{1}{2}\ln|1-x| - \tfrac{1}{2}\ln|1+x| = t + c$$

$$2\ln|x| - \ln|1-x| - \ln|1+x| = 2t + \tilde{c}$$

$$e^{2\ln|x| - \ln|1-x| - \ln|1+x|} = Ae^{2t}$$

$$\left[e^{\ln(x)}\right]^2 e^{-\ln|1-x|}\, e^{-\ln|1+x|} = \frac{\left[e^{\ln|x|}\right]^2}{e^{\ln|1-x|}\, e^{\ln|1+x|}} = \frac{x^2}{(1-x)(1+x)} = Ae^{2t}$$

$$\boxed{\frac{x^2}{(1-x)(1+x)} = Ae^{2t}}$$

implicit soln for $x(t)$.

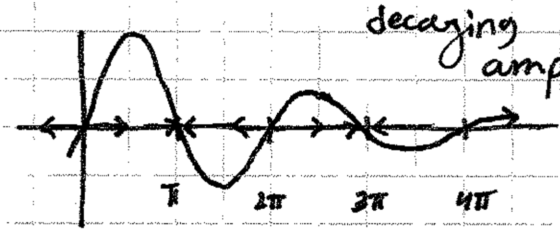
5(d) $\dot{x} = e^{-x}\sin(x)$

decaying exponential this only $\to 0$ when $x \to \infty$

fixed points when $e^{-x}\sin(x) = 0$

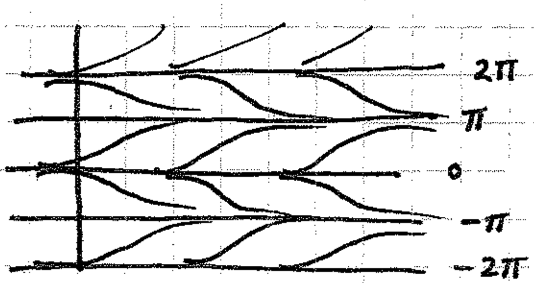
this goes to zero when $x = n\pi$, n = integer.

decaying amplitude



when n = odd fixed point $x = n\pi$ is stable

n = even or zero fixed point $x = n\pi$ is unstable



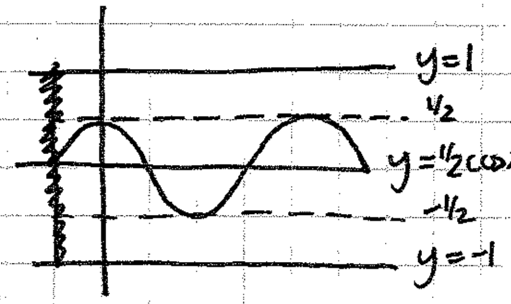
Typical solution curves.

$$\frac{dx}{dt} = e^{-x}\sin(x) \xrightarrow{\text{seperable}} \frac{dx}{e^{-x}\sin(x)} = dt$$

$$\int \frac{e^{x}}{\sin x}\,dx = \int dt$$

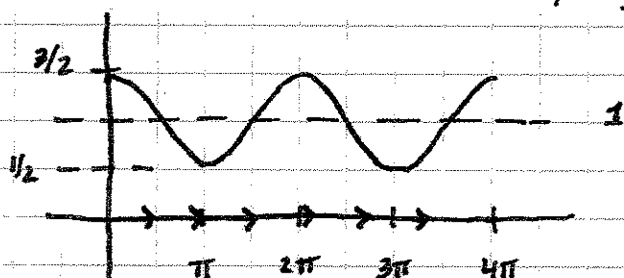
can't integrate this
need hypergeometric functions to evaluate YIKES

5(e) $\dot{x} = 1 + \frac{1}{2}\cos x$ $\quad 1 + \frac{1}{2}\cos(x) = 0$

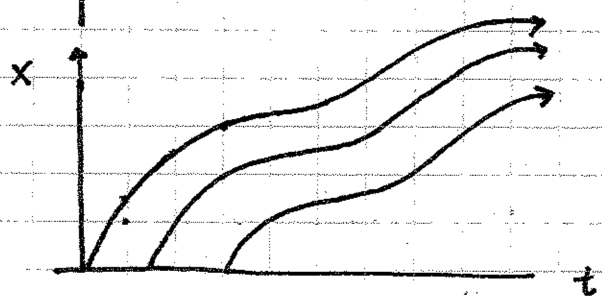


when does $-1 = \frac{1}{2}\cos x$? Never!

there are no fixed points for this system.



always positive flow on the line with changing velocity



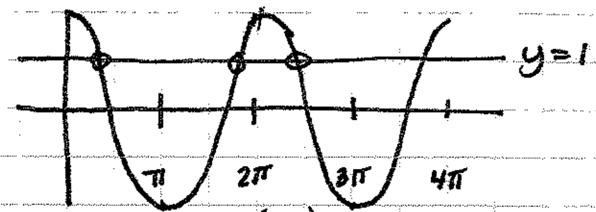
always increasing.

can't find analytical soln $\int \frac{1}{1+\frac{1}{2}\cos x}\,dx = \int dt$

could do trig sub and get result as arctan(tan( ))

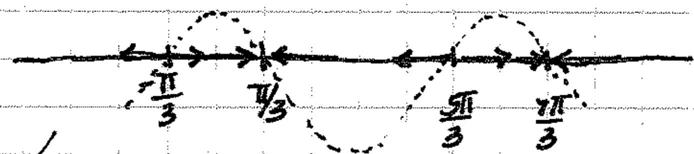
5(f) $\dot{x} = 1 - 2\cos x$

Solve for intersections.



when does $1 - 2\cos x = 0$

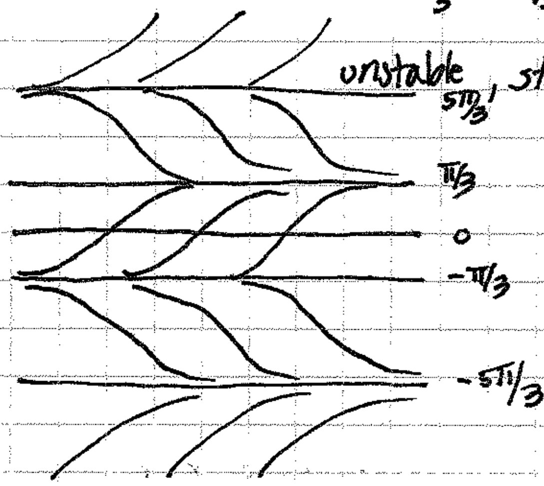
$2\cos x = 1$ or $\cos x = \frac{1}{2}$ @ $x = \arccos(\frac{1}{2}) = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$



flow on a line.

unstable, stable, unstable, stable

stable / unstable points should be reversed what is indicated here



Typical Soln Curves

$$\frac{dx}{dt} = 1 - 2\cos x$$

seperable

$$\int \frac{1}{1 - 2\cos x}\, dx = \int dt$$

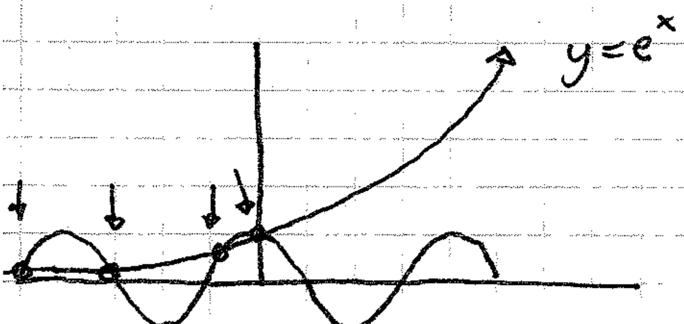
↓

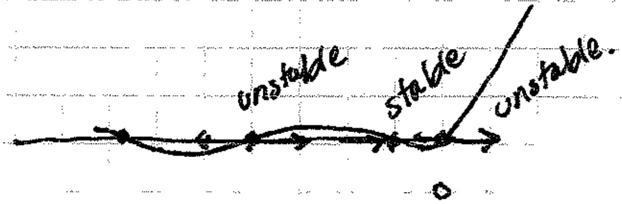
not easily integrated.

6. $\dot{x} = e^x - \cos x$

$e^x - \cos x = 0$

$\cos x = e^x$ draw intersections





negative fixed points @ intersections.

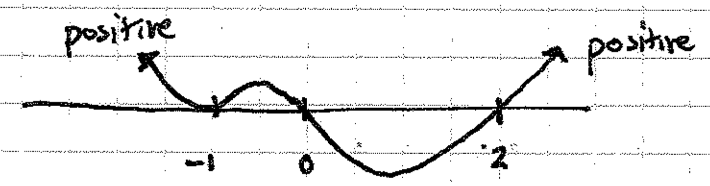
no positive fixed points

7. From flows to equations.

bistable point @ $x=-1$
stable point @ $x=0$
unstable point @ $x=2$

zeros @ $x=-1,0,2$

$(x+1)=0 \quad x=0 \quad (x-2)=0$

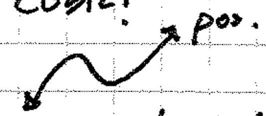


a function like this would work.

so try $x(x+1)(x-2)=0$

↓

cubic!



wont work!

need a 4^{th} order polynomial or some othe function.

TRY $x(x+1)^2(x-2)=0$

↑

bistable point

positive on either side of $x=-1$

test points: $x=-2 \rightarrow -2(-1)^2(-4) = -2(-4) = 8$ positive ✓

$x=-\frac{1}{2} \rightarrow -\frac{1}{2}(-\frac{1}{2})^2(-\frac{5}{2}) =$ positive ✓

$x=1 \rightarrow 1(2)^2(-1) =$ negative ✓

$x=3 \rightarrow 3(4)^2(1) -$ positive ✓

it works!

$$x(x+1)^2(x-2) = x[x^2+2x+1](x-2) = (x^3+2x^2+x)(x-2)$$

$$= x^4 + \cancel{2x^3} + x^2 - \cancel{2x^3} - 4x^2 - 2x$$

$$= x^4 - 3x^2 - 2x$$

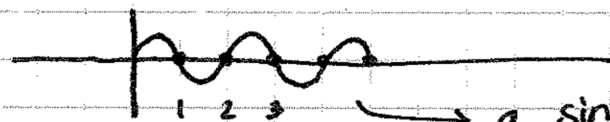
one eqn that would work: $\underline{\dot{x} = x^4 - 3x^2 - 2x}$

8. a) every real # is a fixed point

$x=0 \Rightarrow \boxed{\dot{x}=0}$

every point along x-axis is a fixed point

b) Every integer is a fixed point



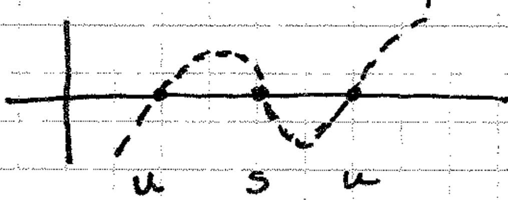
→ a sine or cosine would do this...

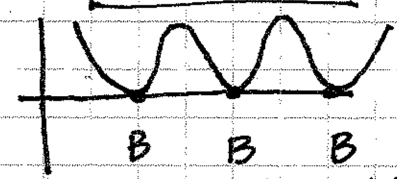
$\sin(n\pi)=0$

$$\boxed{\dot{x}=\sin(x\pi)}$$

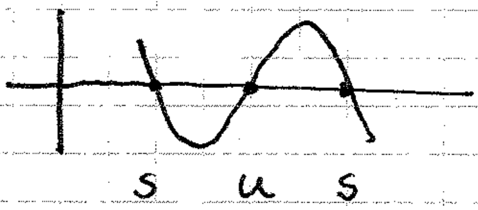
c) Three stable fixed points → $\boxed{\text{Impossible!}}$

There is always an unstable point between two stable points

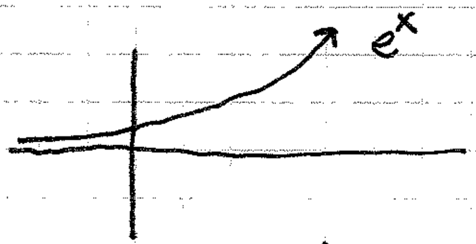




all bistable!!



d) No fixed points → any f(x) that does not cross x-axis



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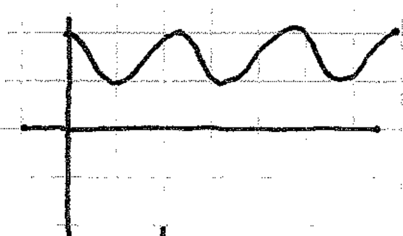
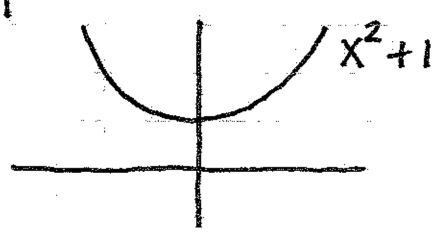


$\boxed{\dot{x}=1}$

or



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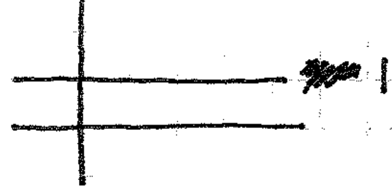


e) There are precisely 100 fixed points.

need a 100^{th} order polynomial w/ 100 real roots or a greater that 100^{th} order polynomial with some repeated roots.

ex

$$\dot{x} = (x+1)(x+2)(x+3)(x+4) \cdots\cdots (x+99)(x+100)$$

9. Analytical Solution Charging Capacitor.

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC} \quad \text{w/ } Q(0)=0$$

$$\frac{dQ}{dt} = \frac{1}{R}\left[V_0 - \frac{Q}{C}\right] \quad \text{seperable ODE.}$$

$$\int \frac{1}{V_0 - Q/C}\, dQ = \int \frac{1}{R}\, dt = \frac{t}{R} + B$$

solve

$$-C \ln\left|V_0 - Q/C\right| = \frac{t}{R} + B$$

$$\ln\left|V_0 - Q/C\right| = \frac{-t}{CR} + \tilde{B}$$

$$V_0 - Q/C = A e^{-t/CR}$$

$$Q = CV_0 - \breve{A} e^{-t/CR}$$

now apply Initial Condition $Q(0) = CV_0 - \breve{A} = 0$

$$\breve{A} = CV_0$$

$$Q = CV_0\left(1 - e^{-t/CR}\right)$$

1. Exact soln to Population Eqn

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) \quad \underline{\text{so}} \quad \frac{dN}{N(1-N/K)} = r\,dt$$

Partial Fractions $\quad \dfrac{1}{N(1-N/K)} = \dfrac{A}{N} + \dfrac{B}{1-N/K}$

$$1 = A - \frac{AN}{K} + BN = \left(B - \frac{A}{K}\right)N + A$$

$$A = 1 \qquad B - 1/K = 0 \qquad B = 1/K$$

$$\int \frac{1}{N}\,dN + \int \frac{1/K}{1-N/K}\,dN = \int r\,dt$$

$$\int \frac{1}{N}\,dN - \int \frac{1/K}{N/K-1}\,dN = \int r\,dt$$

$$\ln|N| - \ln|N/K - 1| = rt + C$$

$$\ln\left|\frac{N}{N/K-1}\right| = rt + c \qquad \frac{N}{N/K-1} = Ae^{rt}$$

$$N(0) = N_0 \qquad \frac{N_0}{N_0/K - 1} = A$$

Solve for N:

$$N = Ae^{rt}(N/K - 1) = \frac{AN}{K}e^{rt} - Ae^{rt}$$

$$N - \frac{A}{K}e^{rt}N = -Ae^{rt} \qquad N = \frac{-Ae^{rt}}{1-\frac{A}{K}e^{rt}} = \frac{1}{\frac{-1}{A}e^{-rt} + 1/K}$$

$$= \frac{AK}{A - Ke^{-rt}}$$

$$\boxed{N = \frac{AK}{A - Ke^{-rt}} \qquad A = \frac{N_0}{N_0/K - 1}}$$

b. $\dot{N} = rN(1 - N/K)$ change variables

$x = 1/N$ so $\dot{x} = \frac{-1}{N^2}\dot{N}$ $\quad \dot{N} = -N^2\dot{x}$

$-N^2\dot{x} = rN(1 - N/K)$ $\quad \dot{x} = \frac{-r}{N}(1 - N/K) = -r\left(\frac{1}{N} - \frac{1}{K}\right)$

but $x = 1/N$

so $\dot{x} = -r(x - 1/K)$

then $\frac{dx}{x - 1/K} = -r\,dt$ $\quad \ln|x - 1/K| = -rt + c$

$x - 1/K = Ae^{-rt}$ $\quad x = Ae^{-rt} + 1/K$

Change back

$\frac{1}{N} = Ae^{-rt} + 1/K$ $\quad N = \frac{1}{Ae^{-rt} + 1/K} = \frac{K}{AKe^{-rt} + 1}$

then $N(0) = \frac{K}{AK + 1} = N_0$

$K = N_0AK + N_0$ or $A = \frac{K - N_0}{N_0K}$

$$\boxed{N = \frac{K}{AKe^{-rt} + 1} \quad A = \frac{K - N_0}{N_0K}}$$

note this A is different from the firs one.

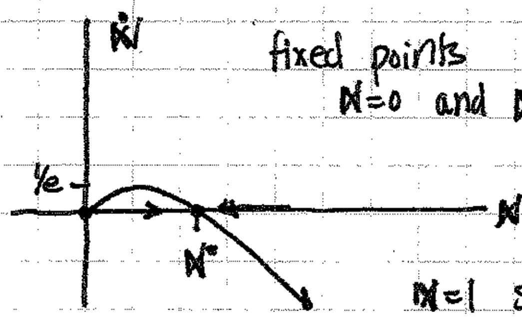
note w/ enough algebra we can make these solutions look identical.

2. Tumor Growth $\dot{N} = -aN\ln(bN)$ 13

Draw $\dot{N}$ vs $f(N)$ a few values of a and b

how to answer part a) ???

$a = b = 1$

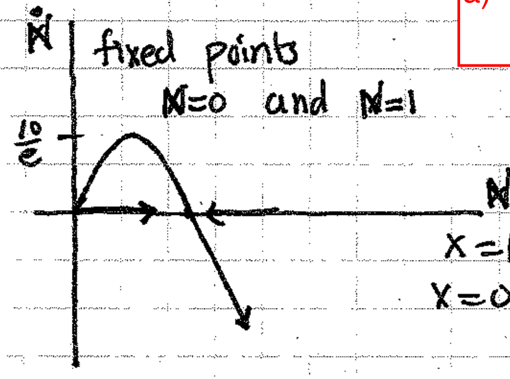


fixed points $N=0$ and $N=1$

$N=1$ stable fixed point

$N=0$ unstable

$a=10$ $b=1$

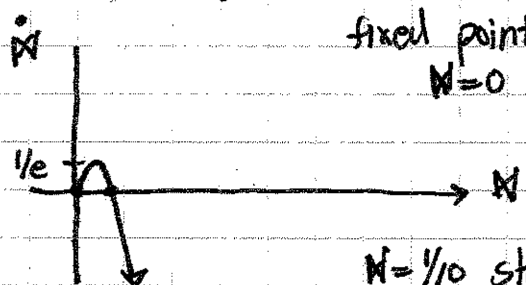


fixed points $N=0$ and $N=1$

$x=1$ stable

$x=0$ unstable

$a=1$ $b=10$



fixed points $N=0$ and $N=1/10$

$N=1/10$ stable

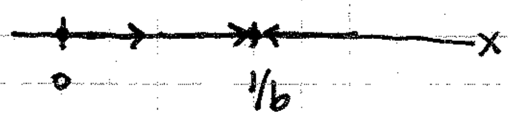
$N=0$ unstable

So changing a changes the maximum for $\dot{N}$

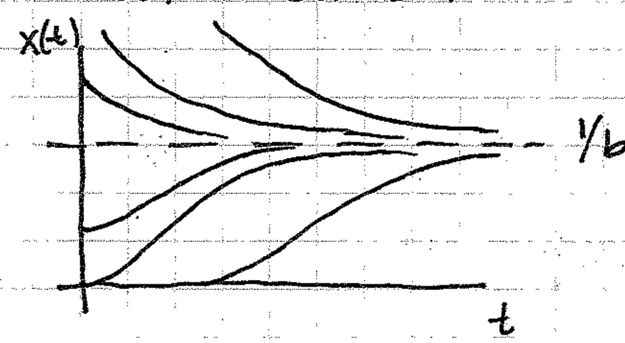
a determines the maximum growth of the tumor

b changes the location of the stable fixed point. Or the stable # of cells, N, in the tumor.

VECTOR FIELD



SOLUTION CURVES.



Linear Stability Analysis.

1. $\dot{x} = x(1-x)$ Find fixed points $x=0$ and $x=1$

consider a small push using the formula

$$\dot{n} = nf'(x) \quad \text{ie. if } \begin{array}{ll} f'(x) > 0 & \text{unstable} \\ f'(x) < 0 & \text{stable.} \end{array}$$

$$f'(x) = (1-x) + x(-1) = 1-2x$$

$$f'(0) = 1 > 0 \quad \text{and} \quad f'(1) = -1 < 0$$

$x=0$ is unstable and $x=1$ is stable.

ie → our small push grows away from $x=0$.

ie → our small push returns or decreases back to the point $x=1$

2. $\dot{x} = x(1-x)(2-x)$ Fixed Points: $x=0$ $x=1$ and $x=2$

$$= (x-x^2)(2-x) = x - 2x^2 - x^2 + x^3 = x^3 - 3x^2 + x$$

find $f'(x) = 3x^2 - 6x + 1$

then $f'(0) = +1 > 0$ ~~stable~~ unstable

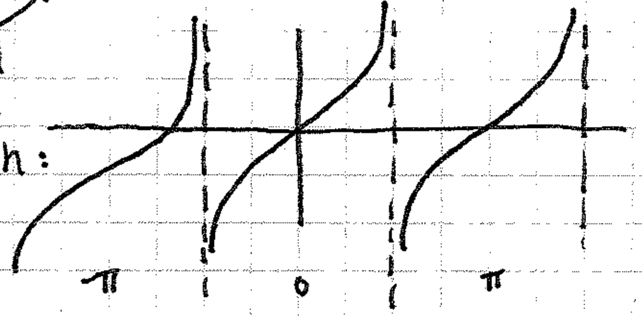
$f'(1) = 3 - 6 + 1 = -2 < 0$ stable

$f'(2) = 12 - 12 + 1 = 1 > 0$ unstable

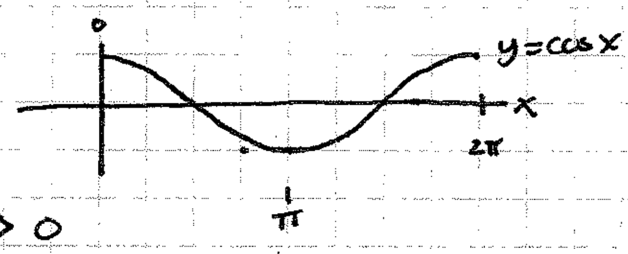
$x=1$ is stable and $x=0$, $x=2$ are unstable

3. $\dot{x} = \tan(x)$ fixed points at $x=0$ $x = \pm\pi$ $x = \pm 2\pi$

read from graph:



then $f'(x) = \dfrac{1}{\cos^2 x}$



$f'(0) = \dfrac{1}{(1)^2} = 1 > 0$

$f'(\pm\pi) = \dfrac{1}{(-1)^2} = +1 > 0$

all fixed points are unstable!

This is ok, because our solution cannot cross the asymptotes of tan(x) function so really if our initial condition is between $-\pi/2 \le x \le \pi/2$

then our only critical point is unstable at $x=0$.

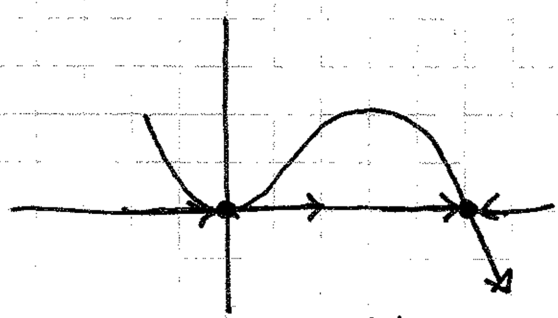
4. $\dot{x} = x^2(6-x)$ fixed points $x=0$ and $x=6$

$\dot{x} = 6x^2 - x^3$ $f'(x) = 12x - 3x^2$

$f'(0)=0$ so this tells us nothing

$f'(6) = 12\cdot 6 - 3\cdot 36 = 72 - 108 < 0$ stable

look at graph

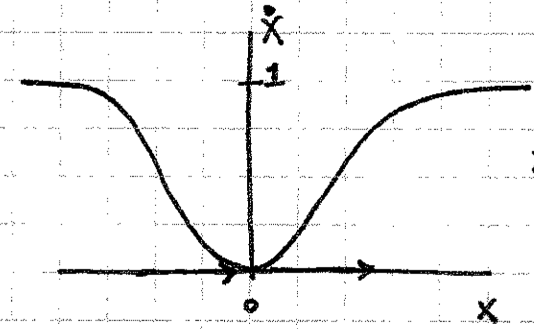


x=0 is bi stable!

5. $\dot{x} = 1 - e^{-x^2}$ fixed point $e^{-x^2} = 1$ when $x = 0$

$$f'(x) = e^{-x^2} \cdot (-2x) = -2xe^{-x^2}$$

$f'(0) = 0$ so we must look @ the graph.



$x=0$ is a bistable fixed point.

6. $\dot{x} = \ln(x)$ $x=1$ is a fixed point

$f'(x) = \frac{1}{x}$ so $f'(1) = \frac{1}{1} = 1 > 0$ unstable

$x=1$ is an unstable fixed point.

we will ignore $x=-1$ as a fixed point because it contains both a real and imaginary part.

7. $\dot{x} = ax - x^3 = x(a - x^2)$ $x=0$ and $x = \pm\sqrt{a}$

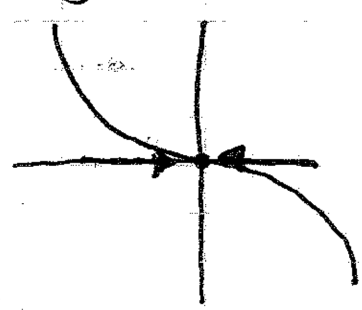
if $a =$ positive $f'(x) = a - 3x^2$

then $f(0) = a > 0$ unstable
$f(\pm\sqrt{a}) = a - 3a < 0$ stable

if $a = 0$ then $f'(x) = a$ and $x=0$ is the only critical point

$f'(0) = 0$ because $a=0$

so we would need a plot of $f(x) = -x^3$



$x=0$ is stable

if $a =$ negative $f(0) = a < 0$ stable and

$$f'(\pm\sqrt{a}) = a - 3a > 0 \quad x = \pm\sqrt{a} \text{ unstable.}$$
$$\text{for } a<0$$

So the stability of the critical points depends of the valve of a!

$a > 0$	$a = 0$	$a < 0$
$x=0$ unstable $x=\pm\sqrt{a}$ stable	$x=0$ stable	$x=0$ stable $x=\pm\sqrt{a}$ unstable.

Existence and Uniqueness.

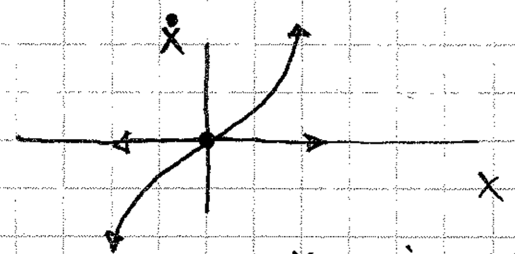
1. $\dot{x} = rx + x^3 \qquad r > 0$

Check existence - uniqueness

$$\left.\begin{aligned} f(x) &= rx + x^3 \\ f'(x) &= r + 3x^2 \end{aligned}\right\} \text{ Solutions exist and are unique } \forall\ x \in \mathbb{R}$$

Solve for the critical points: $rx + x^3 = 0$
$x(r + x^2) = 0$
$x = 0 \quad x = \pm\sqrt{-r}$
but $r > 0$ so these are imaginary.

FLOW ON A LINE



$x=0$ is an unstable fixed point. $\longrightarrow$ so if $x(0) = x_0$ and $x_0 \neq 0$ the solution will go to infinity

To show finite time ... solve:

$$\frac{dx}{x(r+x^2)} = dt \qquad \text{PARTIAL FRACTIONS} \quad \frac{1}{x(r+x^2)} = \frac{A}{x} + \frac{Bx + C}{r + x^2}$$

$$1 = (A+B)x^2 + Cx + Ar$$
$$A = 1/r \quad C = 0 \quad B = -1/r$$

$$\int \frac{1}{x} - \frac{x}{x^2 + r}\, dx = rt + C$$

$$\ln|x| - \tfrac{1}{2}\ln|x^2 + r| = rt + C \quad \Rightarrow \quad \frac{x}{\sqrt{x^2+r}} = Ae^{rt}$$

solve for x: square both sides $\quad \dfrac{x^2}{x^2 + r} = Ae^{2rt}$

$$\Rightarrow \quad x^2 = \frac{rAe^{2rt}}{1 - Ae^{2rt}} \qquad x(0) = x_0 \Rightarrow A = \frac{x_0^2}{r + x_0^2}$$

if $x_0 = 0$ then $A = 0$ and $x = 0$ is soln for all time.

otherwise when denom $\to 0$
$x \to \infty$ this happens when

$$t = \frac{1}{2r}\ln\left|\frac{r + x_0^2}{x_0^2}\right| \quad \text{finite time.}$$

Potentials.

① $\dot{x} = x(1-x)$

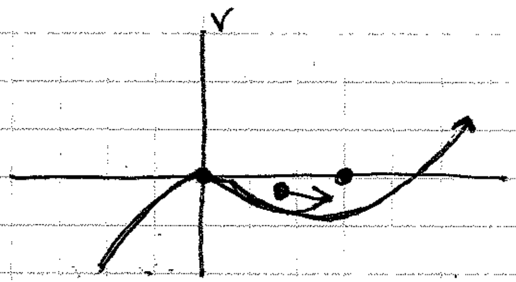
FIXED POINTS: $x(1-x) = 0 \quad x=0 \quad x=1$

POTENTIAL: $-\frac{dV}{dx} = x(1-x) \quad V(x) = -\int x(1-x)\,dx$

$= -\int x - x^2\,dx = \int x^2 - x\,dx$

$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + c \qquad \underline{c=0 \text{ always}}$

$V(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2$



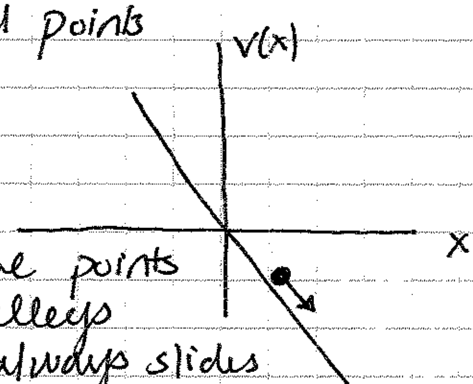
plot V(x) vs x imagine a particle...

$x=0$ is unstable

$x=1$ is stable.

② $\dot{x} = 3$ FIXED POINTS No fixed points

POTENTIAL $V(x) = -\int 3\,dx = -3x$



no critical points

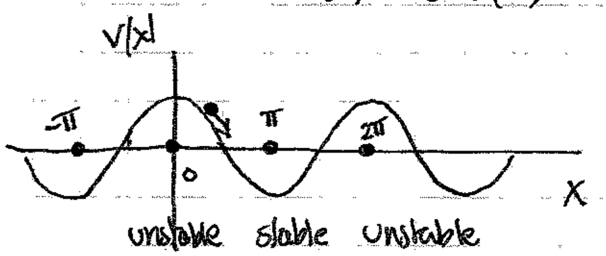
ie. no valleys

particle always slides negative.

③ $\dot{x} = \sin(x)$ FIXED POINTS $x = n\pi \quad n = \text{integer}$

POTENTIAL $V(x) = -\int \sin x\,dx = \cos(x) + c$

$V(x) = \cos(x)$



$n\pi$, n = even unstable

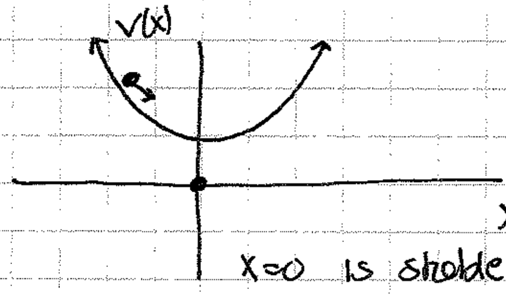
$n\pi$, n = odd stable

(5). $\dot{x} = -\sinh(x)$ FIXED POINT $x=0$

POTENTIAL $V(x) = -\int -\sinh(x)\,dx = \int \sinh(x)\,dx = \cosh(x) + c$

$c=0$ always

$V(x) = \cosh(x)$



note find the integral and plots on WOLFRAM ALPHA.

$x=0$ is stable.

(6). $\dot{x} = r + x - x^3$

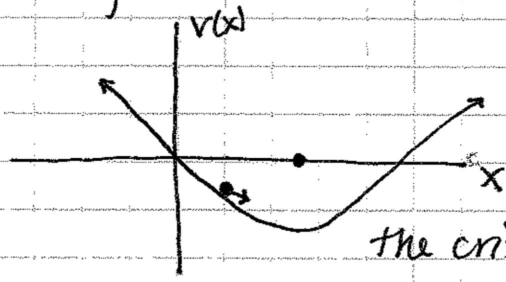
try $r=1$, $r=0$, and $r=-1$

$\boxed{r=1}$ then $\dot{x} = 1 + x - x^3$

This has one real root @ $x \sim 1.3247$.

$$V(x) = -\int 1 + x - x^3\,dx = \int x^3 - x - 1\,dx = \frac{1}{4}x^4 - \frac{1}{2}x^2 - x$$

PLOT $V(x)$ vs X

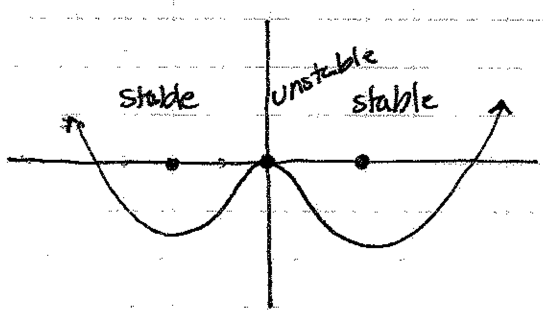


the critical point is stable

$\boxed{r=0}$ then $\dot{x} = x - x^3$ and $x(1-x^2) = 0$ $x = 0, 1, -1$

three critical points

$$V(x) = \int x^3 - x\,dx = \frac{1}{4}x^4 - \frac{1}{2}x^2$$



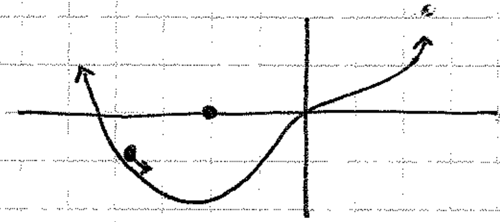
$x=0$ unstable

$x=\pm 1$ stable

$\boxed{r=-1}$ then $\dot{x} = -1 + x - x^3$

one real root @ $x \sim -1.3247$

$$V(x) = \int x^3 - x + 1 \, dx = \tfrac{1}{4}x^4 - \tfrac{1}{2}x^2 + x$$



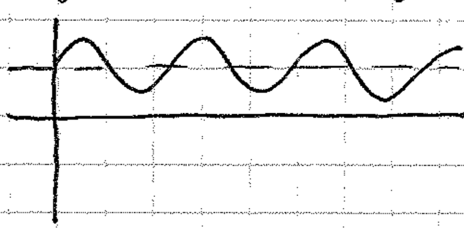
the critical point is stable.

Changing r changes the location and stability of the critical points.

for $|r|$ small $|r| \sim .38$ and smaller the soln has three roots

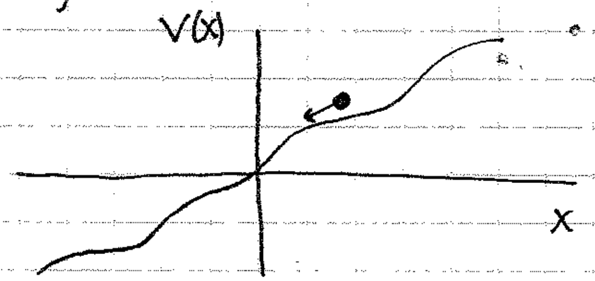
for larger $|r|$ it only has on root.

#4 $\dot{x} = 2 + \sin(x)$



no critical points!

$$V(x) = -\int 2 + \sin x \, dx = -2x + \cos(x)$$



solution goes to the left w/ varying speeds.