Saddle-Node Bifurcations

For each of the following problems, sketch all of the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r, and determine that value. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

1. $\dot{x} = 1 + rx + x^2$
2. $\dot{x} = r - \cosh x$
3. $\dot{x} = r + x - ln(1 + x)$
4. $\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$

Transcritical Bifurcations

For each of the following problems, sketch all of the qualitatively different vector fields that occur as r is varied. Show that a transcritical bifurcation occurs at a critical value of r, and determine that value. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

1. $\dot{x} = rx + \mathbf{x}^2 \longrightarrow$ Typo.
2. $\dot{x} = rx - ln(1 + x)$
3. $\dot{x} = x - rx(1 - x)$
4. $\dot{x} = x(r - e^x)$

Saddle-node Bifurcations – Fixed Points Created or destroyed.

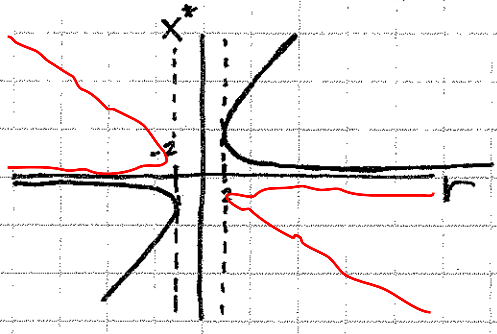
1. $\dot{x} = 1 + rx + x^2$

FIXED POINTS: $x^2 + rx + 1 = 0$ $\quad x = \dfrac{-r \pm \sqrt{r^2-4}}{2}$

CASES: $r^2 - 4 < 0$ imaginary $\quad r^2 - 4 = 0$ one fixed point $\quad r^2 - 4 > 0$ two fixed points

PLOT: $x^2 + rx + 1 = 0$ $\xrightarrow[\text{WAYS}]{\text{TWO}}$ on computer consider $r = -\dfrac{(x^2+1)}{x}$

then flip it!



OR on computer plot $x = \dfrac{-r + \sqrt{r^2-4}}{2}$

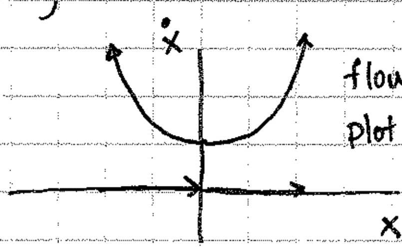
and $x = \dfrac{-r - \sqrt{r^2-4}}{2}$ no flip!

NOW DECIDE STABILITY...

ignore soln w/ imaginary part!

Imaginary: $r^2 - 4 < 0 \Rightarrow r^2 < 4 \Rightarrow r < \pm 2 \Rightarrow -2 < r < 2$

r is inside the range ± 2



flow on line

plot $r = 0$

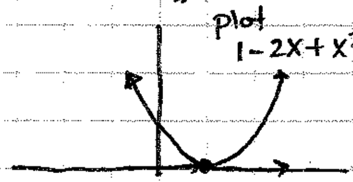
$1 + x^2$

no fixed points always positive flow

One F.P. $r^2 - 4 = 0 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$ consider both...

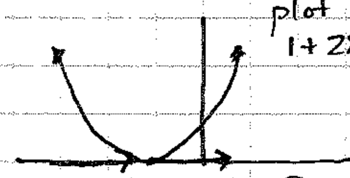
$r = -2$ flow on line $r = 2$

plot $1 - 2x + x^2$



bistable @ $x = 1$

plot $1 + 2x + x^2$



bistable @ $x = -1$

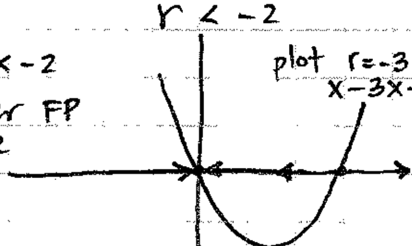
two F.P $r^2 - 4 > 0 \Rightarrow r^2 > 4 \Rightarrow r > \pm 2 \Rightarrow r < -2$ or $r > 2$

$r < -2$ $\quad r > 2$

when $r < -2$ the smaller FP is stable

plot $r = -3$

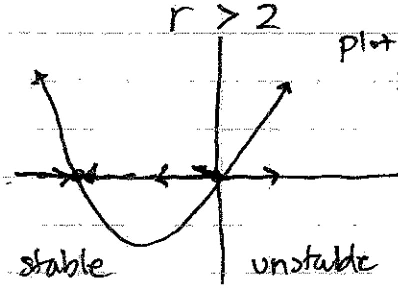
$x - 3x + x^2$



stable unstable

plot $r = 3$

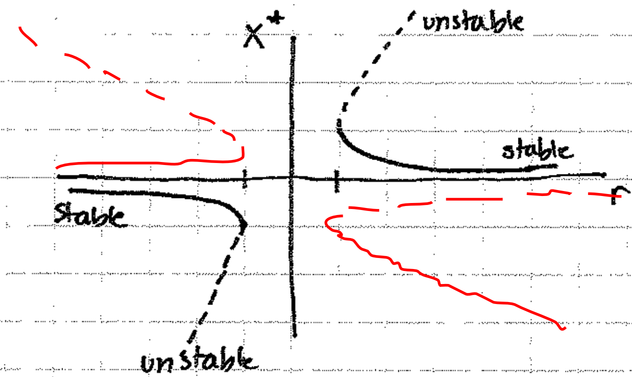
$x + 3x + x^2$



when $r > 2$ the smaller FP is stable

stable unstable

RE-DRAW BIFURCATION DIAGRAM (w/ stability)



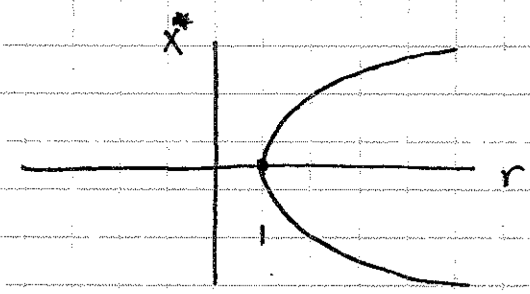
This is like two saddle node bifurcations
Fixed points appear when $r > 2$ or $r < -2$.

BIFURCATION POINTS $r = 2$ and $r = -2$

2. $\dot{x} = r - \cosh(x)$

FIXED POINTS $r - \cosh(x) = 0$ $\quad r = \cosh(x)$ or
$x = \operatorname{arccosh}(r)$

not sure what the cases are... just plot

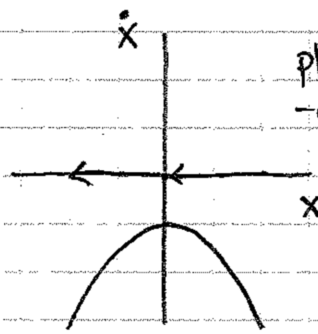


$\cosh(0) = 1$
$x^* = 0$ when $r = 1$

looks like $r = 1$ is a bifurcation point

Now we can see cases: $r < 1$, $r = 1$ and $r > 1$
no FP, one FP and two FP

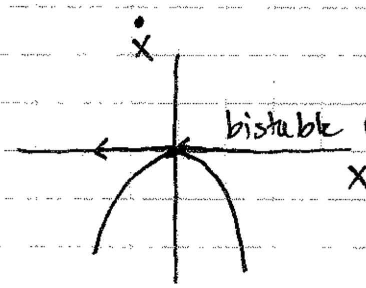
$r < 1$ flow on line



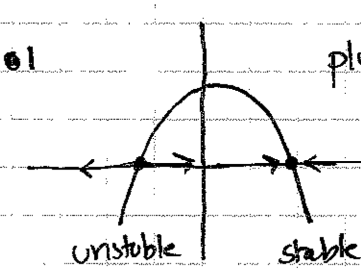
plot $r = 0$
$-\cosh(x)$

no fixed points
flow always negative

$r = 1$



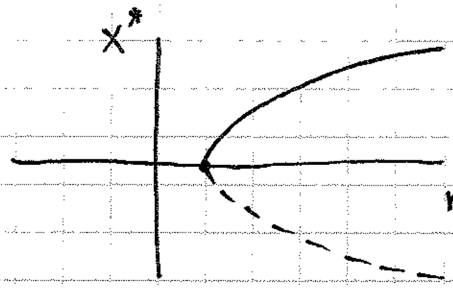
$r > 1$



plot $r = 2$

BIFURCATION POINT $r = 1$

RE-DRAW BIFURCATION (w/ stability)

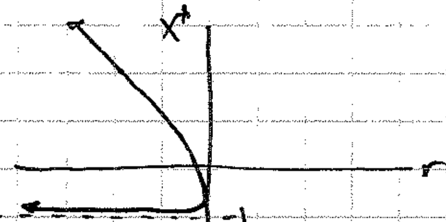


FP's appear when $r>1$
so $r=1$ is a bifurcation point.

3. $\dot{x} = r + x - \ln(1+x)$

FIXED POINTS $r + x - \ln(1+x)$

PLOT x^* vs r



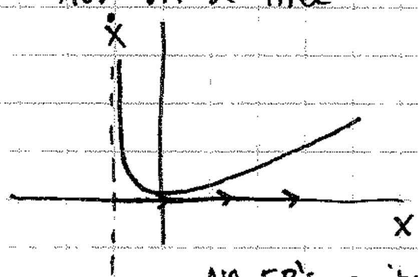
soln become imaginary when $x<-1$

looks like $r=0$ is a bifurcation point

cases: $r>0$, $r=0$, $r<0$
NO FP's, ONE FP, TWO FP's

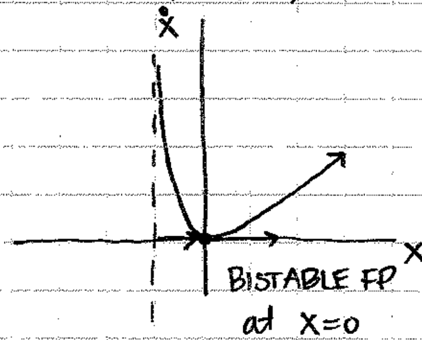
$r>0$ plot $r=1$, $1+x-\ln(1+x)$

flow on a line



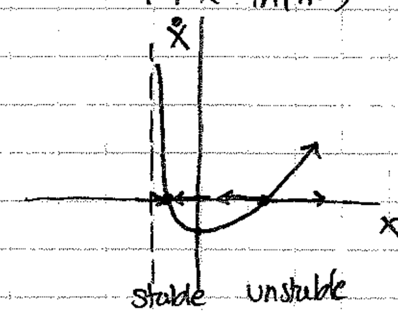
NO FP's positive flow

$r=0$ plot $x-\ln(1+x)$



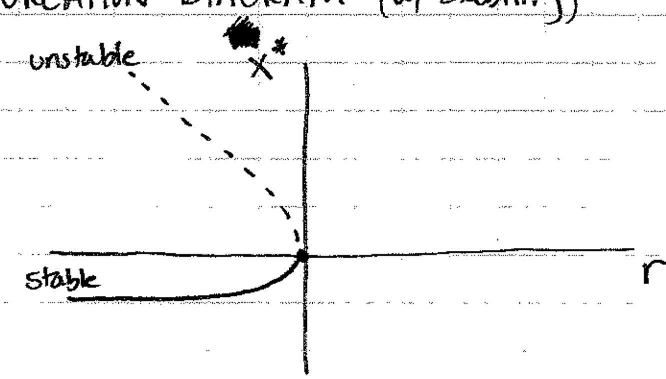
BISTABLE FP at $x=0$

$r<0$ plot $r=-1$ $-1+x-\ln(1+x)$



stable unstable

RE-DRAW BIFURCATION DIAGRAM (w/ stability)

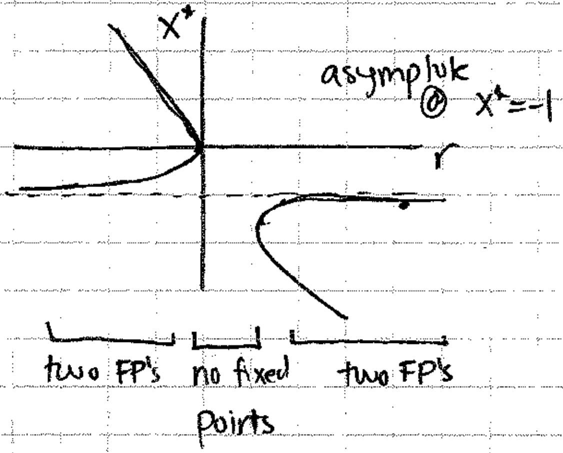


Bifurcation point @ $r=0$

4. $\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$

FIXED POINTS $\quad r + \frac{1}{2}x - \frac{x}{1+x} = 0 \qquad r = \frac{x}{1+x} - \frac{1}{2}x$

PLOT x^* vs r



we can solve for x^*:

$$r(1+x) + \tfrac{1}{2}x(1+x) - x = 0$$
$$2r(1+x) + x(1+x) - 2x = 0$$
$$2r + 2rx + x + x^2 - 2x = 0$$
$$x^2 + (2r-1)x + 2r = 0$$

Use quadratic eqn ...

$$x = \frac{-(2r-1) \pm \sqrt{(2r-1)^2 - 8r}}{2}$$

when $(2r-1)^2 - 8r < 0$ no FP's

$(2r-1)^2 - 8r = 0$ one FP $\longrightarrow 4r^2 - 4r + 1 - 8r = 0$

$(2r-1)^2 - 8r > 0$ two FP's

$$4r^2 - 12r + 1 = 0$$
$$r^2 - 3r + \tfrac{1}{4} = 0$$

CASES: $r < 0.086$

$0.086 < r < 2.914$

$r > 2.914$

$$r = \frac{3 \pm \sqrt{9-1}}{2} = \frac{1}{2}\left[3 \pm \sqrt{8}\right] = \frac{3}{2} \pm \sqrt{2}$$

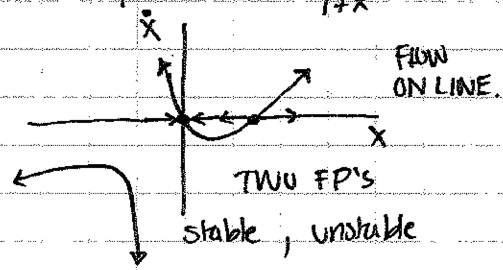
$r \sim 0.086$

$r \sim 2.914$

$r < 0.086$

choose $r = 0$

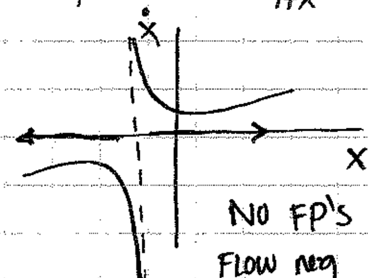
plot $\frac{1}{2}x - \frac{x}{1+x}$



$0.086 < r < 2.914$

choose $r = 1$

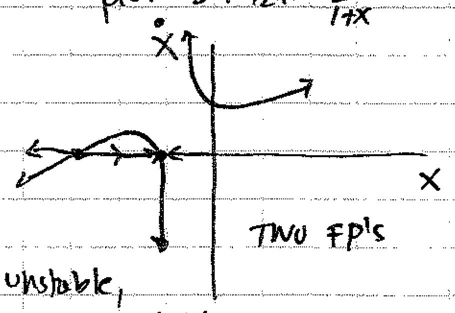
plot $1 + \frac{1}{2}x - \frac{x}{1+x}$



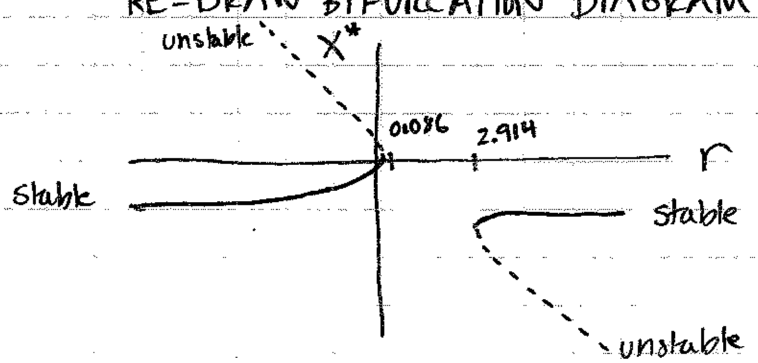
$r > 2.914$

choose $r = 3$

plot $3 + \frac{1}{2}x - \frac{x}{1+x}$



RE-DRAW BIFURCATION DIAGRAM



Bifurcation Points @

$$r = \frac{3}{2} \pm \sqrt{2}$$

Transcritical Bifurcations — Fixed Poins change stability
always at least one FP.

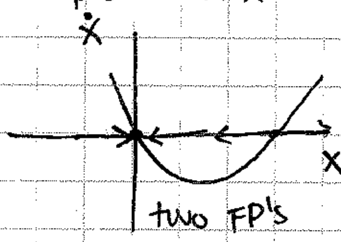
1. $\dot{x} = rx + x^2$ FIXED POINTS $rx + x^2 = 0$

$x(r+x) = 0$

$x=0$ and $x = \cancel{r}\ -r$

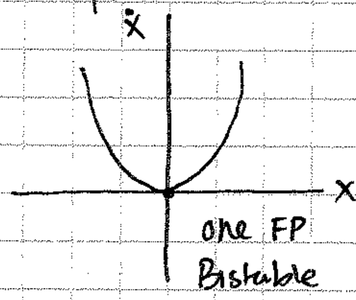
CASES: $r<0$ $r=0$ and $r>0$

$r<0$
choose $r=-1$
plot $-x+x^2$



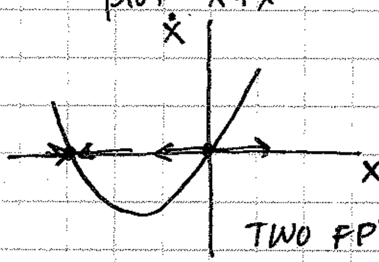
two FP's
stable @ $x=0$
unstable @ $x=r$

$r=0$
plot x^2



one FP
Bistable
@ $x=0$

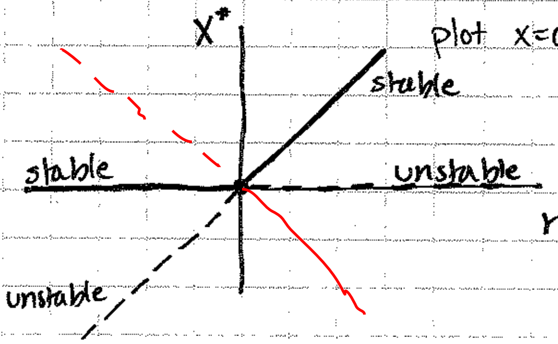
$r>0$
choose $r=1$
plot $x+x^2$



TWO FP's
unstable @ $x=0$
stable @ $x=r$

BIFURCATION DIAGRAM

plot $x=0$ and $x=r$

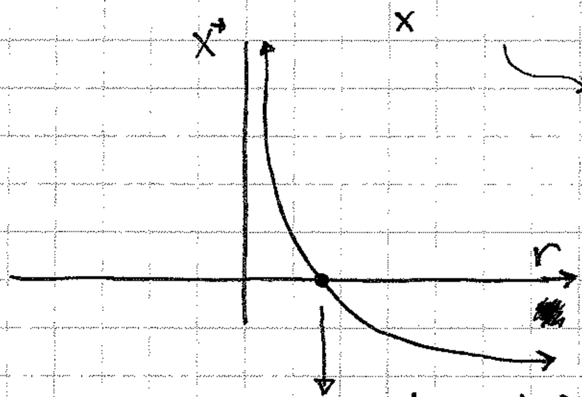


Bifurcation
point @ $r=0$

2. $\dot{x} = rx - \ln(1+x)$

FIXED POINTS $rx - \ln(1+x) = 0$
we see that $x=0$ is always a FP.

PLOT $r = \dfrac{\ln(1+x)}{x}$ on x^* vs r axis.



two asymptotes @ $x=0$ and $x=-1$

$$\lim_{x^*\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{x}\left[x - \tfrac{1}{2}x^2 + O(x^3)\right]$$

$$= \lim_{x\to 0}\left[1 - \tfrac{1}{2}x + O(x^2)\right]$$

$$= 1$$

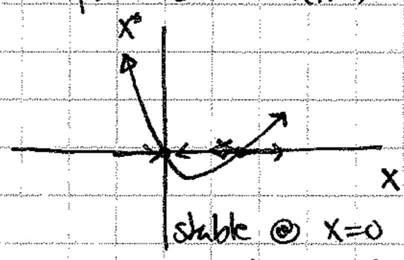
expect a bifurcation where they cross ... @ $r=1$

so $r=1$ is the bifurcation point.

CASES $0 < r < 1$ $r=1$ $r>1$ $r \le 0$

$0 \le r < 1$
choose $r = 1/2$
plot $\tfrac{1}{2}x - \ln(1+x)$

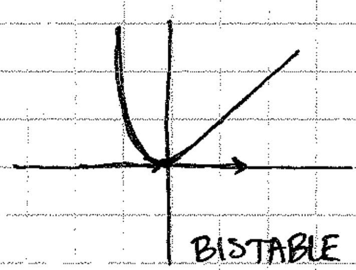
FLOW FIELD!



stable @ $x=0$
unstable @ other FP

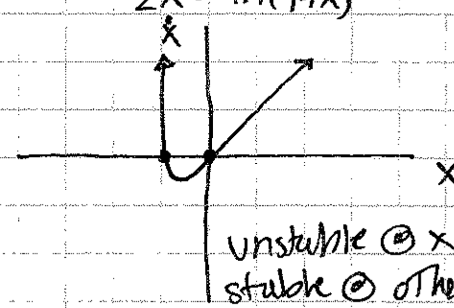
$r=1$
plot $-\ln(1+x$

FLOW FIELD!



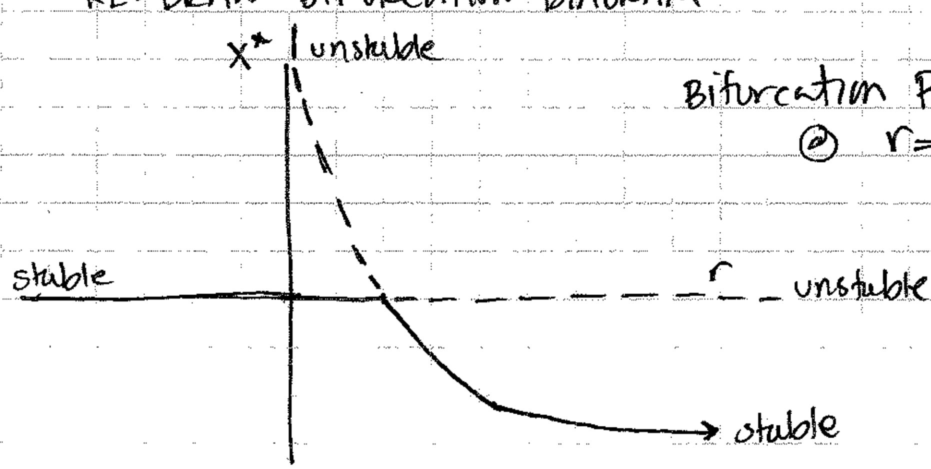
BISTABLE @ $x=0$

$r>1$
choose $r=2$
$2x - \ln(1+x)$



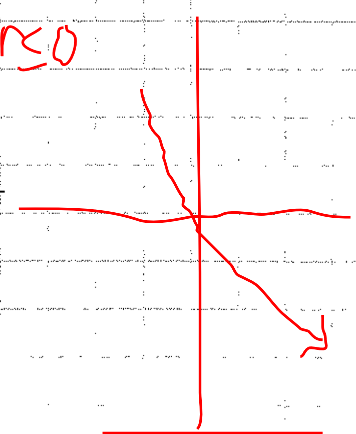
unstable @ $x=0$
stable @ other FP

RE-DRAW BIFURCATION DIAGRAM



Bifurcation Point @ $r=1$

$r \le 0$



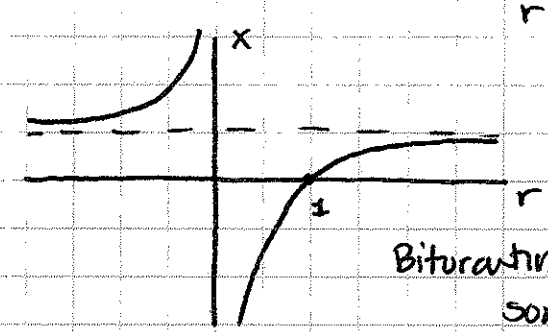
$\boxed{\text{stable @ x=0}}$

3. $\dot{x} = x - rx(1-x)$

FIXED POINTS:

$$x - rx(1-x) = 0$$
$$x(1 - r(1-x)) = 0$$
$$x = 0 \qquad 1 - r(1-x) = 0 \longrightarrow r = \frac{1}{1-x} \rightarrow x \neq 1$$
$$r - rx = 1$$
$$-rx = 1 - r$$
$$x = \frac{r-1}{r} \longrightarrow r \neq 0$$

Plot $x=0$ and $x = \frac{r-1}{r}$



Bifurcation diagram w/ no stability.

Biturcation @ $r=1$

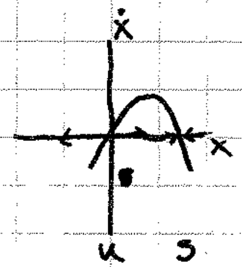
something @ $r=0$

CASES $r<0$, $0<r<1$, $r>1$

FLOW ON LINE

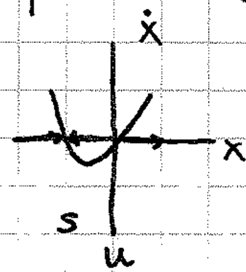
$r<0$ choose $r=-1$

plot $x + x(1-x)$



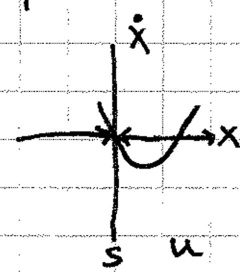
$0<r<1$ choose $r=\frac{1}{2}$

plot $x - \frac{1}{2}x(1-x)$

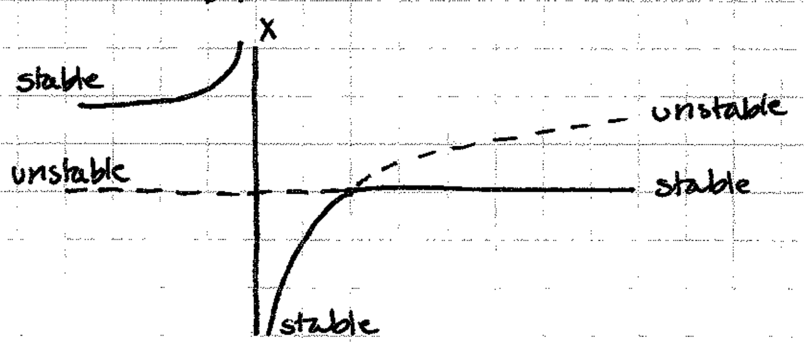


$r>1$ choose $r=2$

plot $x - 2x(1-x)$



BIFURCATION DIAGRAM



Bifurcation At $r=1$

Soln can't cross $r=0$ unless it is on $x=0$.

4. $\dot{x} = x(r - e^x)$

FIXED POINTS $\quad x(r-e^x)=0$

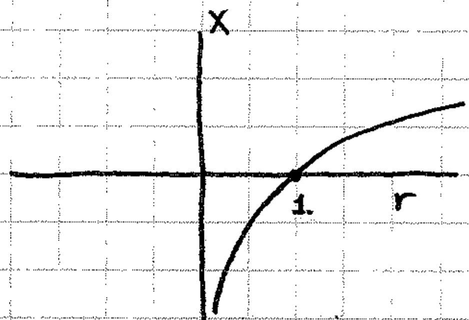
$x=0$ always FP

$r=e^x$ or $x=\ln(r)$

$r<0$ imaginary

$r>0$ real.

Plot $x=0$ and $x=\ln(r)$



Bifurcation point where they cross $r=1$

double check $r<0$

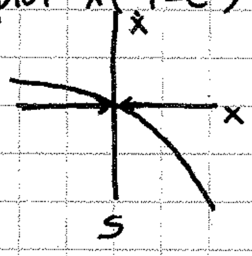
CASES

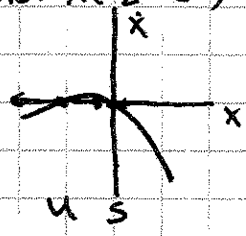
$r<0$ choose $r=-1$ plot $x(-1-e^x)$

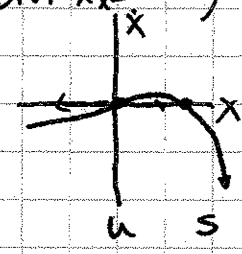
$0<r<1$ choose $r=\frac{1}{2}$ plot $x(\frac{1}{2}-e^x)$

$r>1$ choose $r=2$ plot $x(2-e^x)$

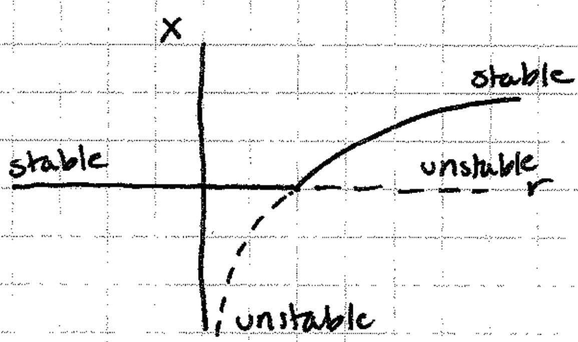
FLOW ON LINE







BIFURCATION DIAGRAM



Bifurcation Point $r=1$

Taylor Series. $\qquad \underset{\text{near } x=a}{f(x)} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + O(x^3)$

1. $f(x) = e^{-x} \qquad n=5$

$f'(x) = -e^{-x} \qquad f''(x) = e^{-x}$

$f'(5) = -e^{-5} \qquad f''(5) = e^{-5} \qquad f(5) = e^{-5}$

$$\underset{\text{near } 5}{f(x)} = e^{-5} - e^{-5}(x-5) + \frac{e^{-5}}{2}(x-5)^2 + O(x^3)$$

2. $f(x) = \sin(x) \qquad n=4$

$f'(x) = \cos(x) \qquad f''(x) = -\sin(x)$

$f'(4) = \cos(4) \qquad f''(4) = -\sin(4) \qquad f(4) = \sin(4)$

$$\underset{\text{near } 4}{f(x)} = \sin(4) + \cos(4)(x-4) - \frac{\sin(4)}{2}(x-4)^2 + O(x^3)$$

3. $f(x) = \frac{1}{1+x} \qquad n=4$

$f'(x) = \frac{+1}{(1+x)^2} \qquad f''(x) = \frac{2}{(1+x)^3}$

$f'(4) = \frac{+1}{(5)^2} \qquad f''(x) = \frac{2}{(5)^3} \qquad f(5) = \frac{1}{5}$

$$\underset{\text{near } 4}{f(x)} = \frac{-1}{3} + \frac{1}{9}(x-4) + \frac{1}{27}(x-4)^2 + O(x^3)$$

4. $f(x) = \sqrt{1+x} \qquad n=3$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad f''(x) = -\frac{1}{4}(1+x)^{-3/2}$

$f'(3) = \frac{1}{2}(4)^{-1/2} \qquad f''(3) = -\frac{1}{4}(4)^{-3/2} \qquad f(3) = \sqrt{4}$

$= \frac{1}{4} \qquad = -\frac{1}{16} \qquad = 2$

$$\underset{\text{near } 3}{f(x)} = 2 + \frac{1}{4}(x-3) - \frac{1}{32}(x-3)^2 + O(x^3)$$

Pitchfork Bifurcations

#1 $\dot{x} = rx + 4x^3$

FIXED POINTS $x(r+4x^2)=0$

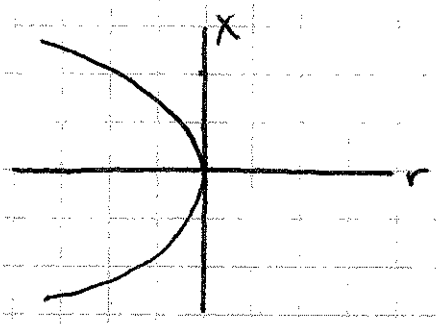
$x=0 \quad r+4x^2=0$

$x^2 = \frac{-r}{4} \qquad x = \frac{\pm\sqrt{-r}}{4}$

$r<0$ real

$r>0$ imaginary.

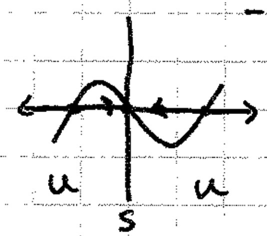
Plot $x=0$ and $x = \frac{\pm\sqrt{-r}}{2}$



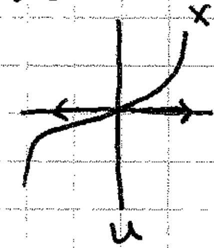
CASES:

FLOW ON LINE

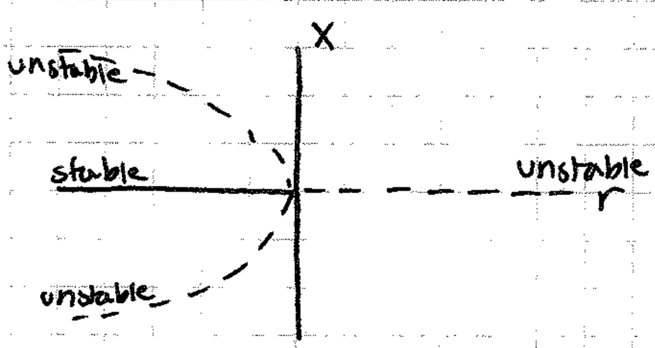
$r<0$ choose $r=-1$
$-x+4x^3$



$r>0$ choose $r=1$
$x+4x^3$



BIFURCATION DIAGRAM



Subcritical Pitchfork
Bifurcation point $r=0$

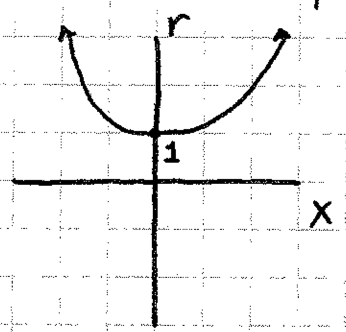
#2 $\dot{x} = rx - \sinh(x)$

FIXED POINTS $rx - \sinh(x) = 0$

$x = 0$ always FP $\sinh(0) = 0$

$$r = \frac{\sinh(x)}{x}$$

PLOT r vs x
Then Flip

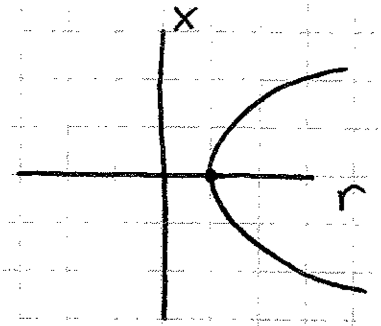


NOTE $\lim_{x \to 0} \frac{\sinh(x)}{x} = 1$

when x large
r positive large

$x = 0$ $r = 1$

x small r positive large

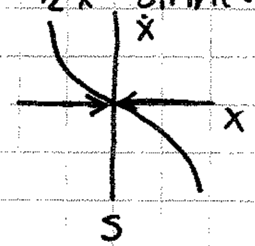


r=1 looks like Bifurcation

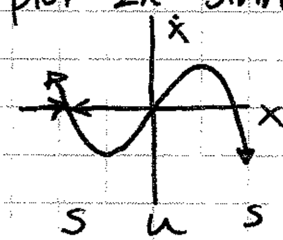
CASES

$r < 1$
choose $r = 1/2$
plot $\frac{1}{2}x - \sinh(x)$

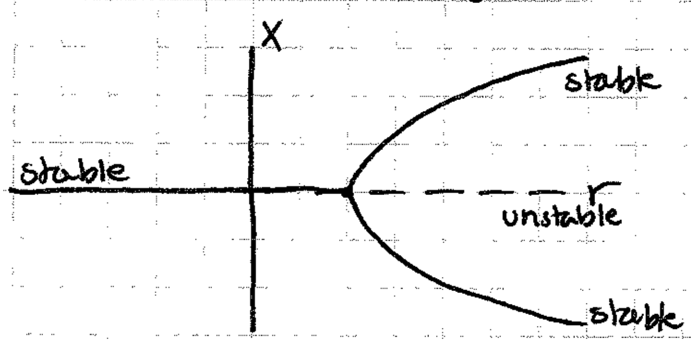
FLOW ON LINE



$r > 1$
choose $r = 2$
plot $2x - \sinh(x)$



BIFURCATION DIAGRAM



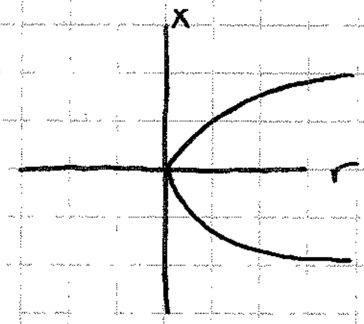
Supercritical Pitchfork Bifurcation @ $r = 1$

3. $\dot{x} = rx - 4x^3$

FIXED POINTS $x(r-4x^2)=0$

$x=0 \quad r-4x^2=0 \quad x = \pm\frac{\sqrt{r}}{2}$

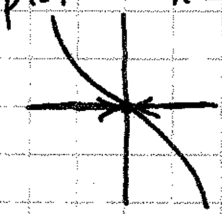
Plot $x=0$ and $x = \pm\frac{\sqrt{r}}{2}$ $r<0$ imag
$r>0$ real.



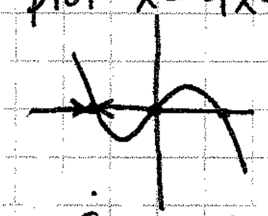
CASES

FLOW ON LINE

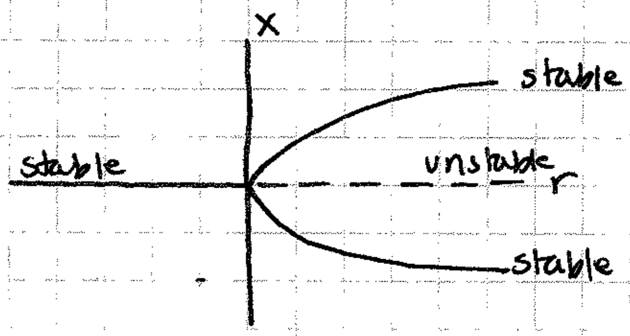
$r<0$
choose $r=-1$
plot $-x-4x^3$



$r>0$
choose $r=1$
plot $x-4x^3$



BIFURCATION DIAGRAM



Supercritical Pitchfork
Bifurcation point
$r=0$

#4 $\dot{x} = x + \frac{rx}{1+x^2}$

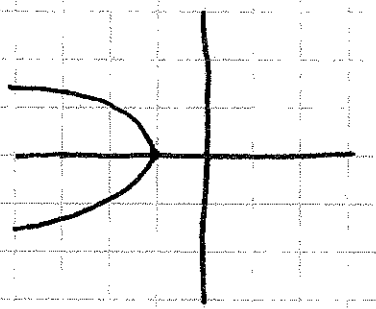
FIXED POINTS $x\left(1 + \frac{r}{1+x^2}\right) = 0$ $x=0$ always FP

$\frac{r}{1+x^2} = -1$ $r = -1 - x^2$ $-r = 1 + x^2$

$x^2 = -(1+r)$ $x = \pm\sqrt{-(1+r)}$

Plot $x=0$ and $x = \pm\sqrt{-(1+r)}$

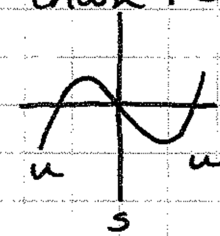
$r < -1$ real
$r > -1$ imag



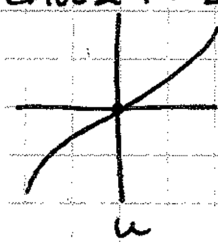
$r=-1$ looks like bifurcation

CASES

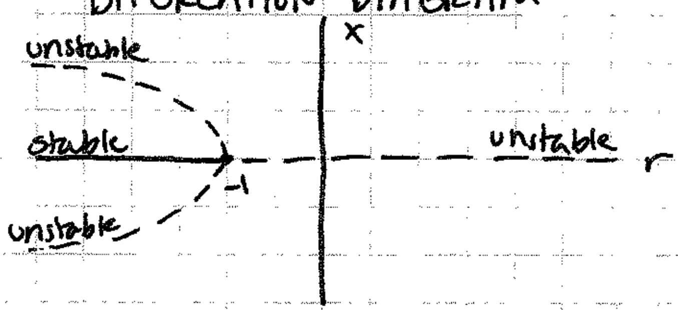
$r < -1$
choose $r = -2$



$r > -1$
choose $r = 2$



BIFURCATION DIAGRAM



Subcritical Pitchfork Bifurcation @ $r = -1$

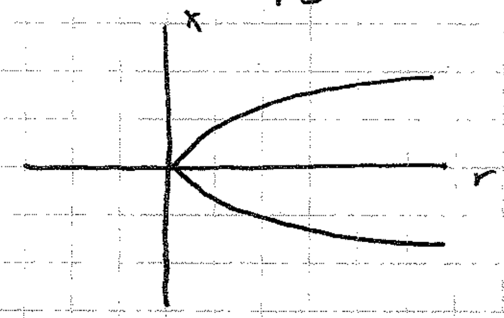
General Bifurations

1. $\dot{x} = r - 3x^2$

FIXED POINTS $\quad r - 3x^2 = 0 \quad x = \pm\sqrt{\frac{r}{3}}$

$r < 0$ imag $\quad r > 0$ real

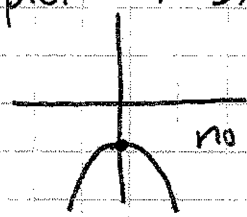
PLOT $\quad x = \pm\sqrt{\frac{r}{3}}$



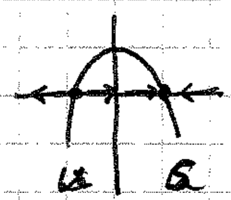
Bifuration Point $r = 0$

CASES

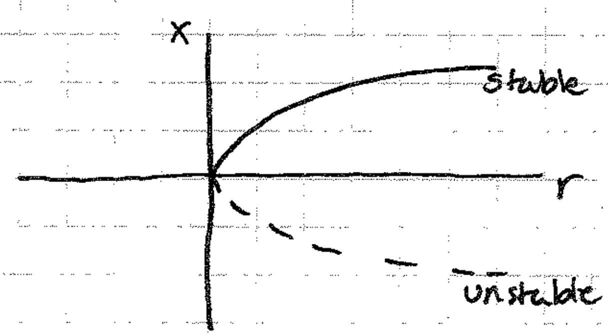
$r < 0$
choose $r = -1$
plot $-1 - 3x^2$



$r > 0$
choose $r = 1$
plot $1 - 3x^2$



BIFURCATION DIAGRAM



Saddle-Node Bifurcation
@ $r = 0$

2. $\dot{x} = 5 - re^{-x^2}$

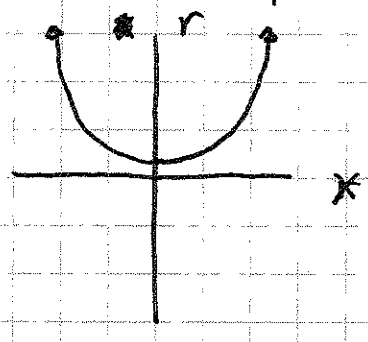
FIXED POINTS $5 - re^{-x^2} = 0$

$e^{-x^2} = \frac{5}{r}$ $\quad -x^2 = \ln\left(\frac{5}{r}\right)$

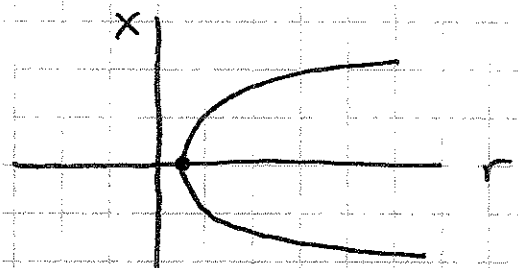
real when $r > 5$

imag when $r < 5$ $\quad x = \pm\sqrt{-\ln\left(\frac{5}{r}\right)}$ → this is complicated!

instead plot $r = 5e^{x^2}$ and flip



FLIP THIS →

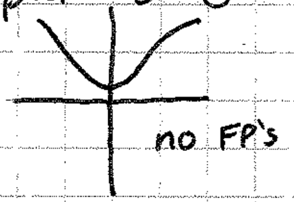


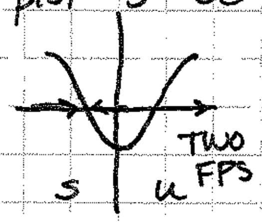
Bifurcation $r = 5$

CASES $\quad r < 5$ $\qquad r > 5$

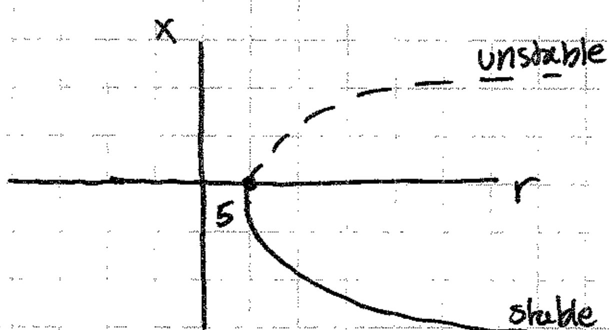
choose $r = 1$ $\qquad$ choose $r = 6$

plot $5 - e^{-x^2}$ $\qquad$ plot $5 - 6e^{-x^2}$





BIFURCATION DIAGRAM



Saddle-node Bifurcation at $r = 5$

3. $\dot{x} = rx - \dfrac{x}{1+x^2}$

FIXED POINTS: $x\left(r - \dfrac{1}{1+x^2}\right) = 0$

$x=0$ always a FP

other FP's $r = \dfrac{1}{1+x^2}$

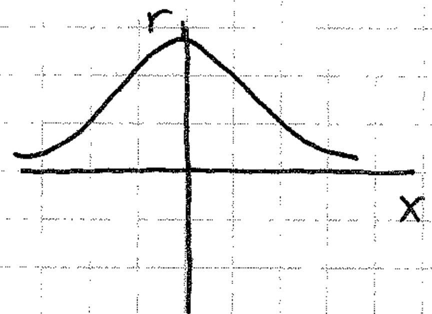
PLOT r vs x then flip

or $1+x^2 = \dfrac{1}{r}$ $x^2 = \dfrac{1}{r} - 1$

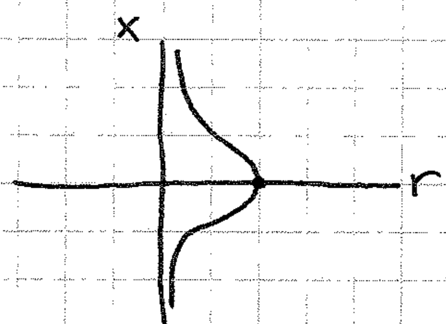
$x = \pm\sqrt{\dfrac{1}{r} - 1}$

$\dfrac{1}{r} > 1$ real $\quad r \neq 0$

$\dfrac{1}{r} < 1$ imag



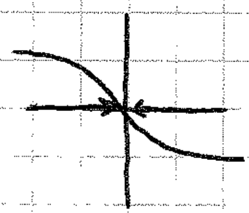
FLIP THIS $\longrightarrow$



CASES $0 > r$

choose $r = -1$

plot $-x - \dfrac{x}{1+x^2}$

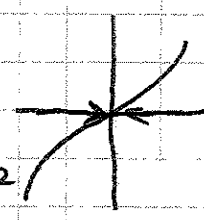


one FP Stable

$1 < r$

choose $r = 2$

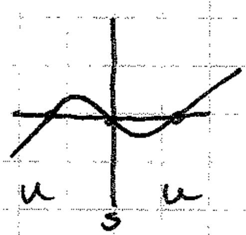
plot $2x - \dfrac{x}{1+x^2}$



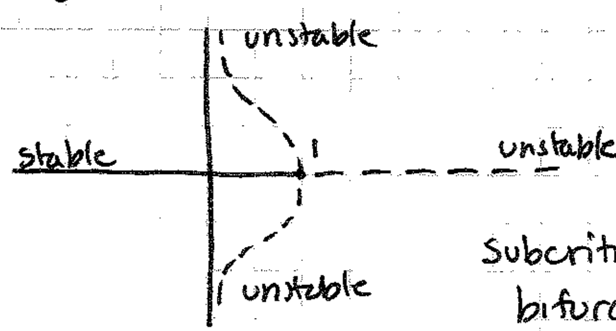
one FP Unstable

$0 < r < 1$

choose $r = 1/2$



BIFURCATION DIAGRAM



Subcritical Pitchfork bifurcation @ $r=1$

4. $\dot{x} = rx - \frac{x}{1+x}$

Fixed points: $x\left(r - \frac{1}{1+x}\right) = 0$ $\quad$ $x=0$ always FP

$r = \frac{1}{1+x} \Rightarrow r + rx = 1$

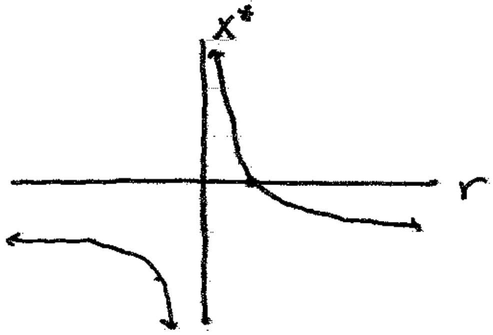
$rx = 1 - r$

$x = \frac{1-r}{r}$ $\quad$ asymptote $r \neq 0$

$x \neq -1$ asymptote

$x = 0$ when $r = 1$

$r=1$ Bifurcation point.

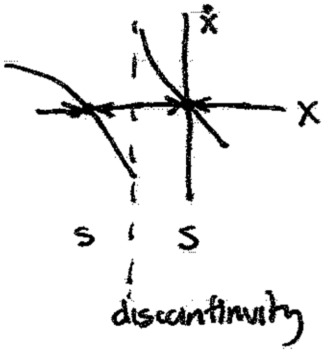


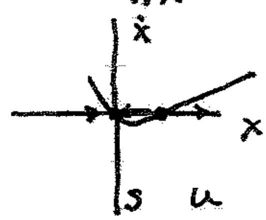
cases $\quad r<0$ $\quad$ $0<r<1$ $\quad$ $r>1$

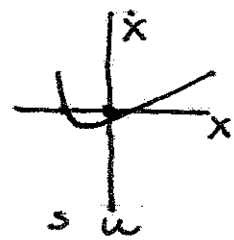
plot $r=-1$ $\quad$ $r=\frac{1}{2}$ $\quad$ $r=2$

$-x - \frac{x}{1+x}$ $\quad$ $\frac{1}{2}x - \frac{x}{1+x}$ $\quad$ $2x - \frac{x}{1+x}$

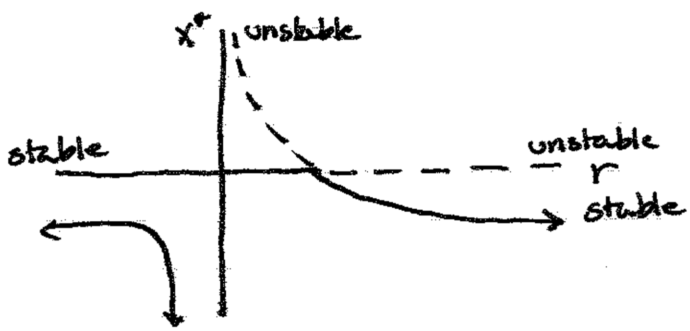
Flow on Line







BIFURCATION DIAGRAM



Transcritical Bifurcation @ $r=1$

Discontinuity @ $x=-1$.