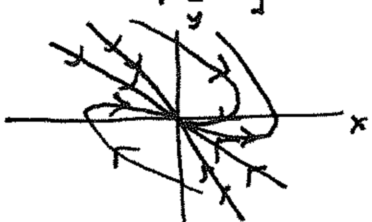
Homework Week 3 Solns.

DAY 1

For the following linear systems: First write the systems in matrix form. Then use Pplane to draw the phase portrait. For each problem classify the type of fixed point using the vocabulary that we developed in class. Discuss what happens to your solution if you start at $(x_0, y_0) = (1, 0)$

1. $\dot{x} = y, \quad \dot{y} = -2x - 3y$
2. $\dot{x} = 5x + 10y, \quad \dot{y} = -x - y$
3. $\dot{x} = 3x - 4y, \quad \dot{y} = x - y$
4. $\dot{x} = -3x + 2y, \quad \dot{y} = x - 2y$
5. $\dot{x} = -y, \quad \dot{y} = -x$

1. $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}$

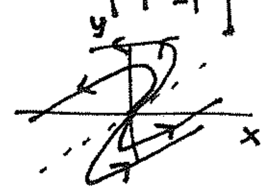


Actually a stable node. looks like a degenerate node if you're not careful.
(1,0) $x, y \to 0, 0$ as $t \to \infty$

2. $\dot{\vec{x}} = \begin{bmatrix} 5 & 10 \\ -1 & -1 \end{bmatrix} \vec{x}$

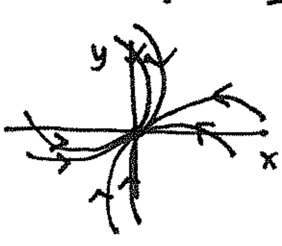
unstable spiral.
(1,0) $x, y \to \infty$ as $t \to \infty$

3. $\dot{\vec{x}} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$



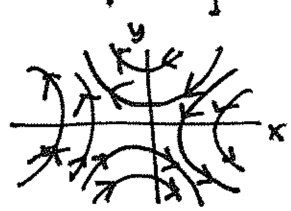
degenerate node unstable.
(1,0) $x, y \to \infty$ as $t \to \infty$

4. $\dot{\vec{x}} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \vec{x}$



stable node
(1,0) $x, y \to 0, 0$ as $t \to \infty$

5. $\dot{\vec{x}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \vec{x}$



Saddle node
(1,0) $x, y \to \infty, -\infty$ as $t \to \infty$

DAY 2

Find the eigenvalues and eigenvectors for the linear systems. Classify the fixed point using the eigenvalues. Compare your results to what you found on Day 1 (these are the same four systems that you should already have written in matrix form).

1. $\dot{x} = y, \quad \dot{y} = -2x - 3y$
2. $\dot{x} = 5x + 10y, \quad \dot{y} = -x - y$
3. $\dot{x} = 3x - 4y, \quad \dot{y} = x - y$
4. $\dot{x} = -3x + 2y, \quad \dot{y} = x - 2y$
5. $\dot{x} = -y, \quad \dot{y} = -x$

For the following linear systems: First write the systems in matrix form. Then Classify the fixed point based on the trace and determinant of the matrix. Finally, find the eigenvalues and eigenvectors for the system and draw a reasonably accurate phase portrait. Discuss what happens to your solution if you start at $(x_0, y_0) = (1, 0)$

6. $\dot{x} = 5x + 2y, \quad \dot{y} = -17x - 5y$
7. $\dot{x} = -3x + 4y, \quad \dot{y} = -2x + 3y$
8. $\dot{x} = 4x - 3y, \quad \dot{y} = 8x - 6y$
9. $\dot{x} = y, \quad \dot{y} = -x - 2y$

Day 2.

1. $\lambda = -1, -2$

should be a stable node

$\lambda = -1 \quad \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = -2 \quad \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

2. $\lambda = \dfrac{2 \pm \sqrt{16-20}}{2} = 2 \pm i$

complex with positive real part... unstable spiral.

Evals complex $\Rightarrow$ Evecs complex.

don't need to solve for them

$\lambda_1 = 2+i$

$V_1 = \begin{bmatrix} -3-i \\ 1 \end{bmatrix}$

$\lambda_2 = 2-i$

$V_2 = \begin{bmatrix} -3+i \\ 1 \end{bmatrix}$

3. $\lambda = 1, 1$

repeated positive e-vals

expect a special case.

unstable degenerate node.

$\lambda = 1 \quad \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad V = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

4. $\lambda = -1, -4$

should be a stable node.

$\lambda = -1 \quad \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \vec{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -4 \quad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

5. $\lambda = -1, 1$

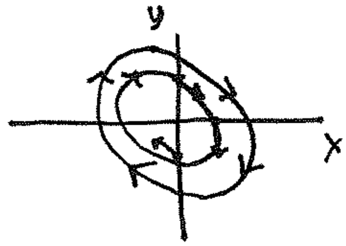
saddle point.

$\lambda = -1 \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 1 \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad V_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

All of these should math what we found on day 1.

6. $\dot{\vec{x}} = \begin{bmatrix} 5 & 2 \\ -17 & -5 \end{bmatrix} \vec{x}$



$\lambda = \pm 3i$

this is a center.

7. $\dot{\vec{x}} = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} \vec{x}$

$\lambda = \pm 1$

saddle point.

$\lambda_1 = -1$

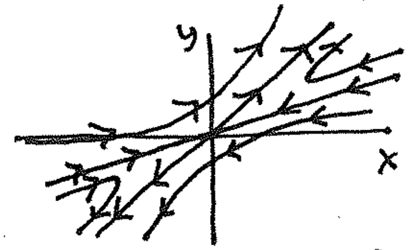
$\begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ stable manifold.

$\lambda_2 = 1$

$\begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ unstable manifold



8. $\dot{\vec{x}} = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \vec{x}$

$\lambda = 0, -2$

star, degenerate, or non unique?

$\lambda_1 = 0$

$\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$V_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

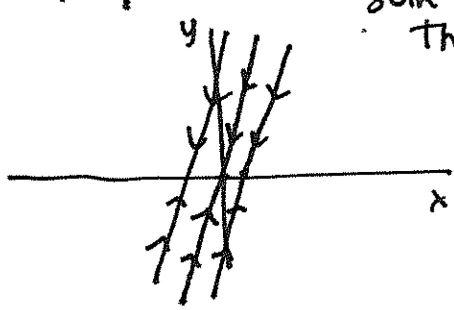
soln constant on this vector

$\lambda = -2$

$\begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

soln decays on this vector.



line of fixed points.

9. $\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \vec{x}$

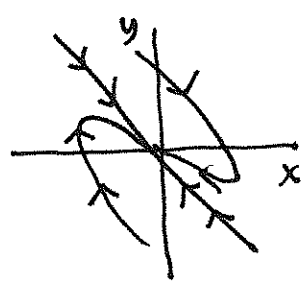
$\lambda = -1$

degenerate node.

$\lambda = -1$

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



DAY 3

Use a computer (aka PPLANE) to plot the phase portrait of the following nonlinear systems. Write in words what you seen in the phase portrait. What different types of solutions might you expect. In each case graphically find the fixed points and discuss their stability based on the phase portrait.

1. $\dot{x} = y, \quad \dot{y} = -x + y(1 - x^2)$ (van der Pol oscillator)

2. $\dot{x} = 2xy, \quad \dot{y} = y^2 - x^2$ (Dipole fixed point)

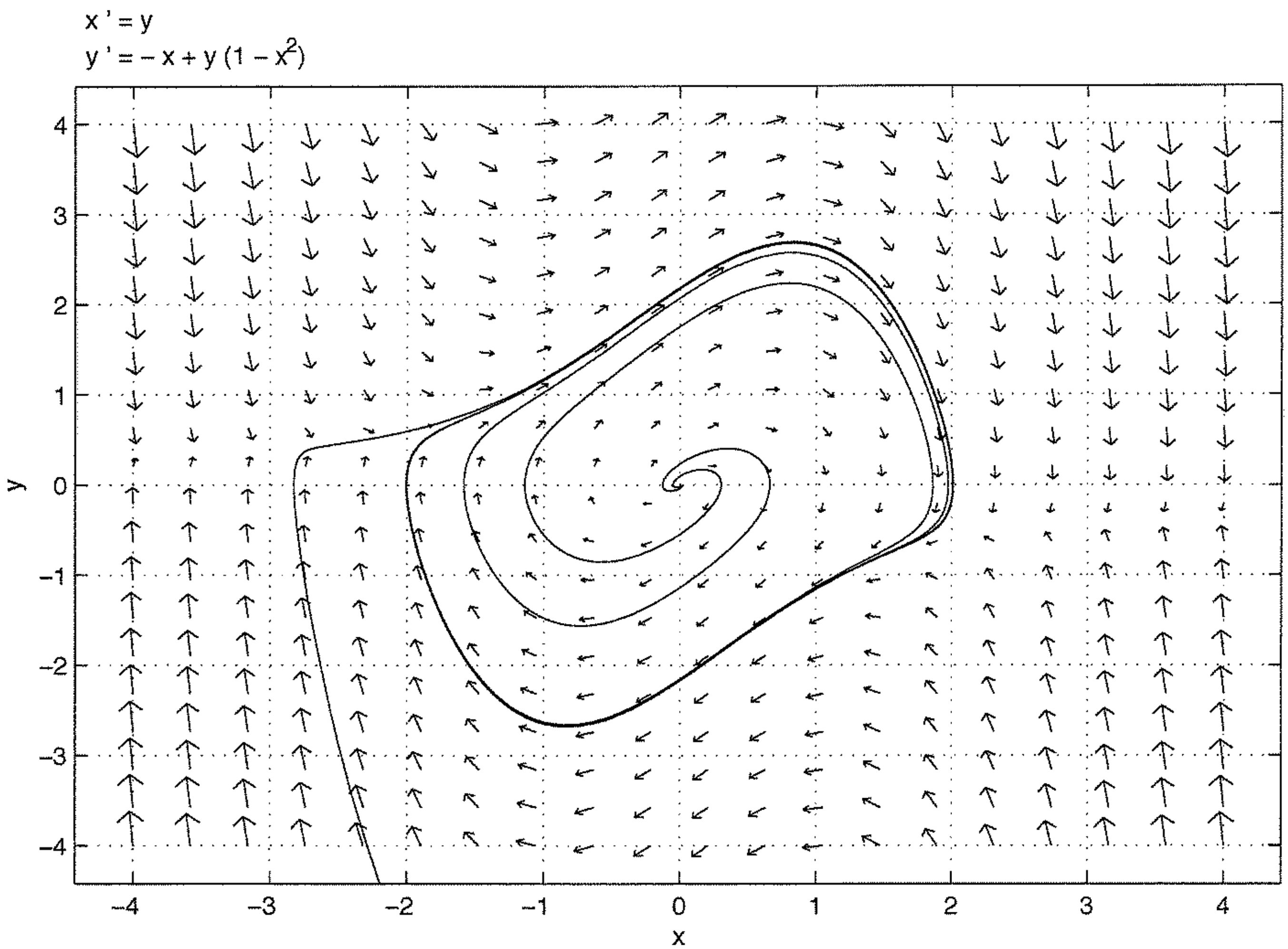
3. $\dot{x} = y + y^2, \quad \dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (Two-eyed monster)

4. $\dot{x} = y + y^2, \quad \dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (Parrot)

For each of the following systems: Find all possible fixed points, linearize the system near these fixed points and classify them, sketch the phase portrait near the fixed points, and finally try to fill in the phase portrait as best you can. Remember to check your answers using Pplane but please attempt to sketch the solution BEFORE checking with Pplane.

5. $\dot{x} = x - y, \quad \dot{y} = x^2 - 4$
6. $\dot{x} = \sin y, \quad \dot{y} = x - x^3$
7. $\dot{x} = y + x - x^3, \quad \dot{y} = -y$
8. $\dot{x} = 1 + y - e^{-x}, \quad \dot{y} = x^3 - y$

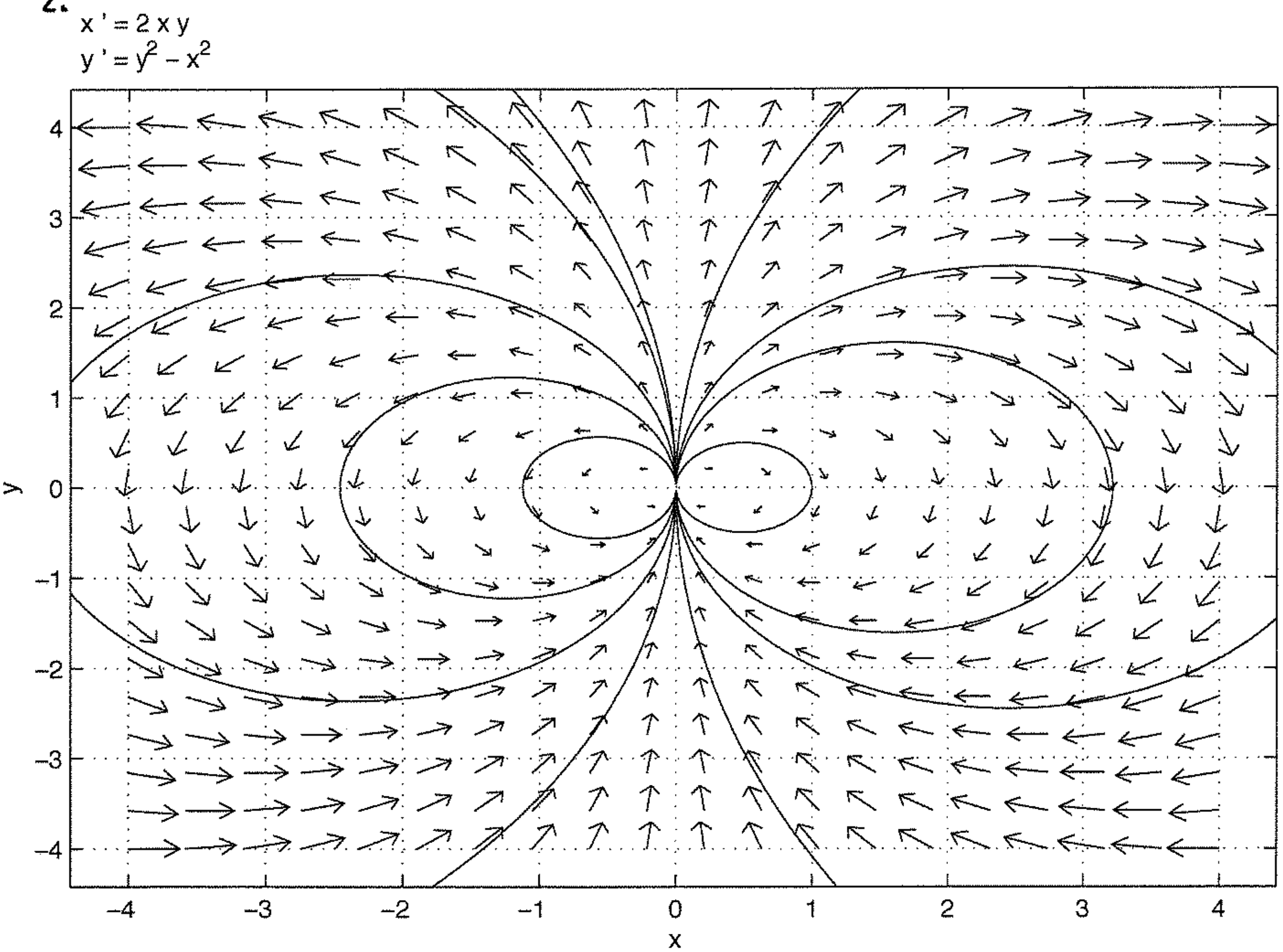
1.



Has a closed orbit. The fixed point is an unstable spiral.

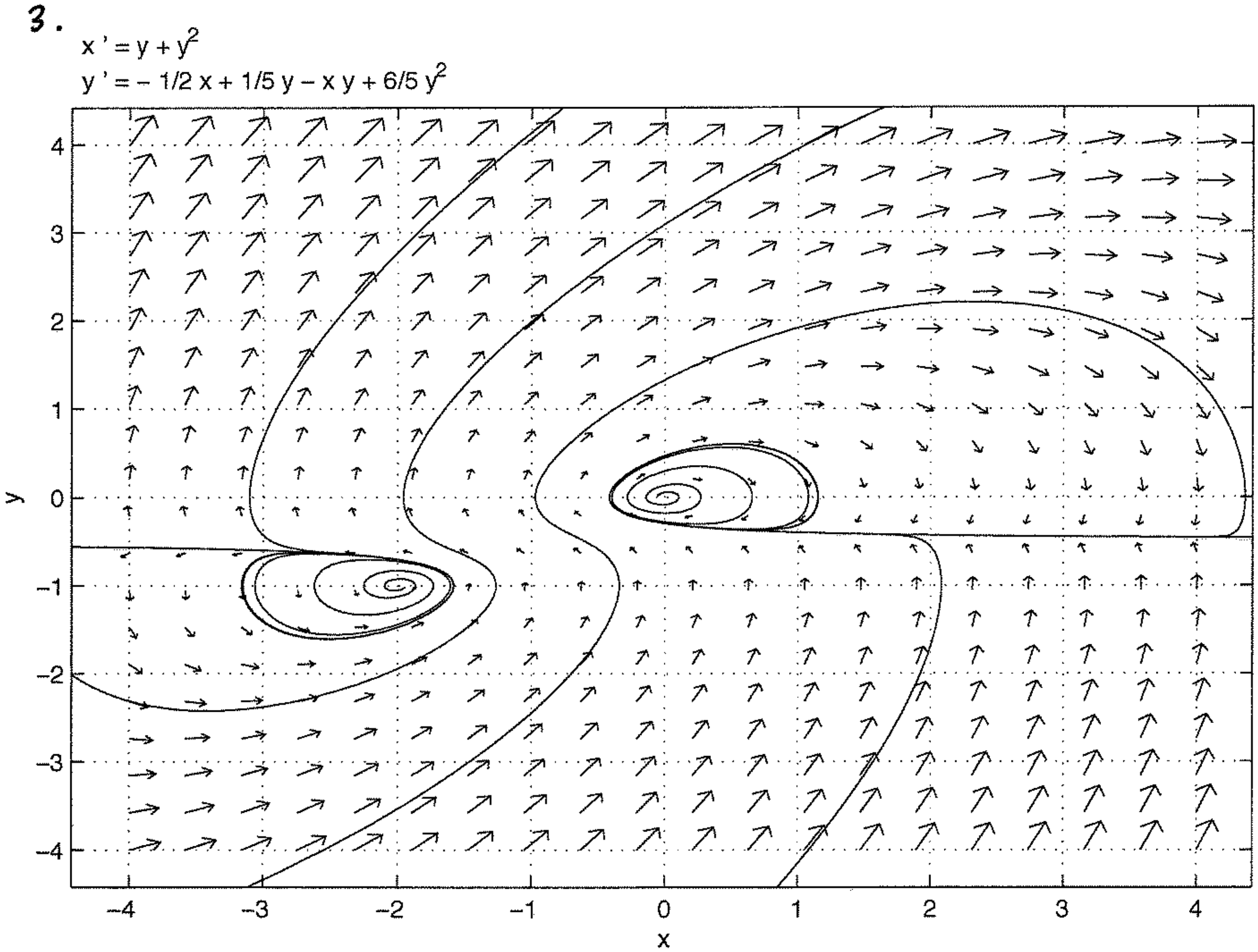
The fixed point is liaponov stable since trajectories that start close stay close.

2.



Wherever we start we return to
the fixed point at (0,0)
we may have to travel far away first.

3.



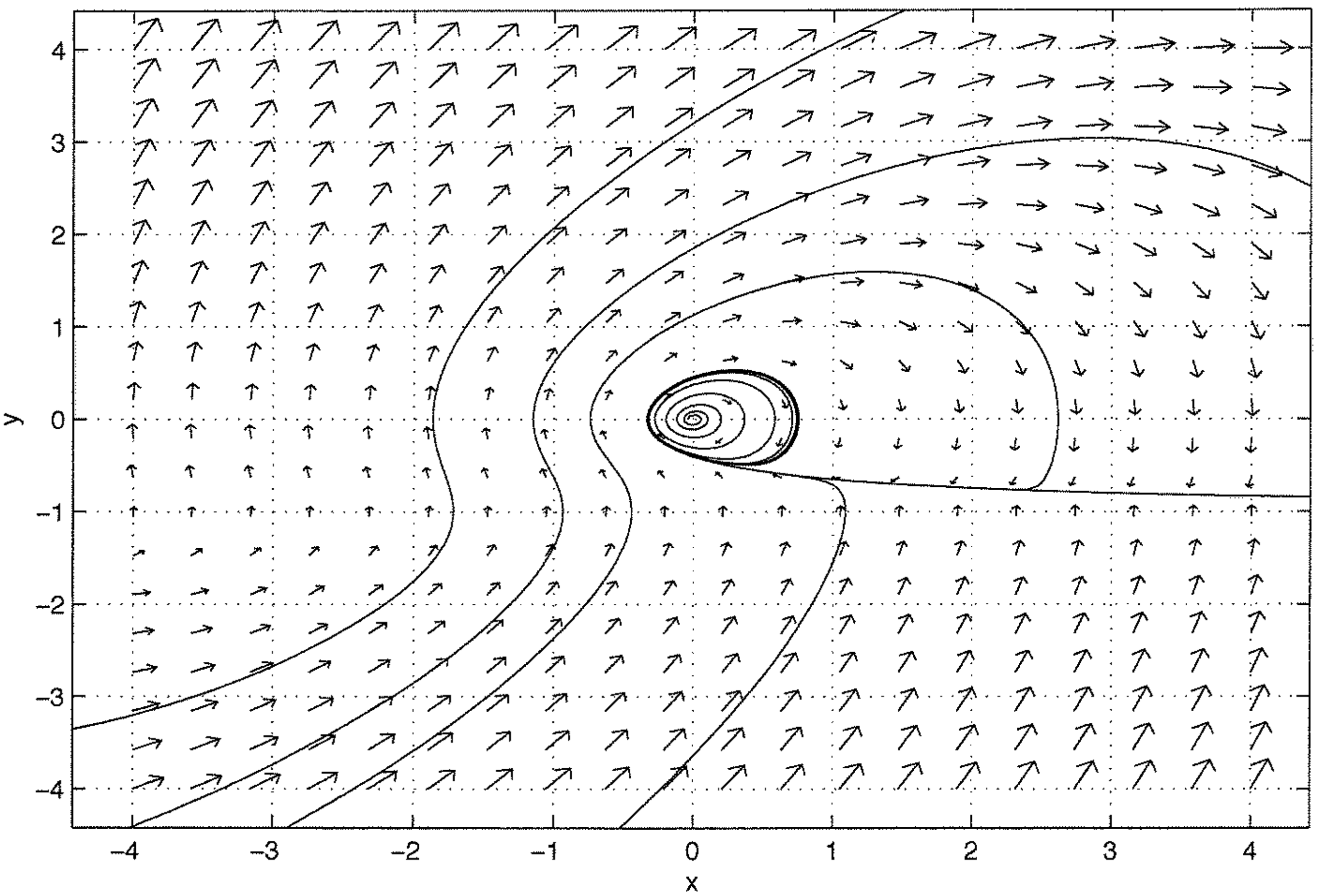
Two fixed points (0,0) and (-2,-1)

unstable spiral
solns ~~[illegible]~~ go toward the orbit.

stable spiral.
not attracting because if we are outside the orbit the soln goes toward the orbit around (0,0).

4. $x' = y + y^2$

$y' = -x + 1/5\, y - x\, y + 6/5\, y^2$



(0,0) is a stable spiral

Start outside the orbit — solns are attracted to the orbit.

Start inside the orbit — solns are attracted to (0,0).

nullclines: $y=x$ $\updownarrow$

$x=\pm 2$ $\leftrightarrow$

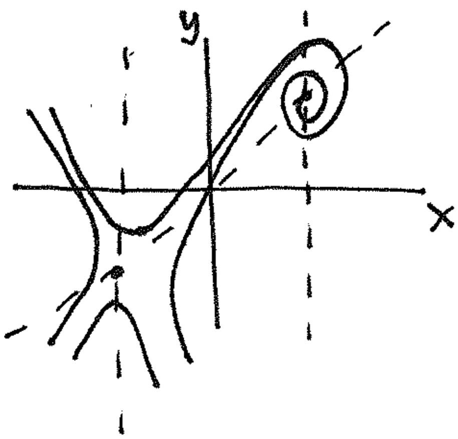
5. $\dot{x} = x-y$

$\dot{y} = x^2-4$

two fixed points $(-2,-2)$ $(2,2)$.

$J = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}$ $\quad J\Big|_{(-2,-2)} = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}$ $\quad \lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{17}$

this is a saddle.



$J\Big|_{(2,2)} = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$. $\quad \lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{15}\, i$

unstable spiral.

nullclines: $y=n\pi$ $\updownarrow$

$x=0,-1,1$ $\leftrightarrow$

6. $\dot{x} = \sin(y)$

$\dot{y} = x - x^3$

infinite # of fixed points $\quad x = 0, 1, -1$, $y = n\pi$ $\rightarrow$ $(0, n\pi)$, $(1, n\pi)$, $(-1, n\pi)$

$J = \begin{bmatrix} 0 & \cos(y) \\ 1-3x^2 & 0 \end{bmatrix}$

$J\Big|_{(0,n\pi)} = \begin{bmatrix} 0 & \cos(n\pi) \\ 1 & 0 \end{bmatrix}$

$J\Big|_{(1,n\pi)} = \begin{bmatrix} 0 & \cos(n\pi) \\ -2 & 0 \end{bmatrix}$

$n=$ even

$J\Big|_{(0,n\pi)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\lambda = \pm 1$

saddle nodes

$J\Big|_{(1,n\pi)} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

$\lambda = \pm 2i$

centers

$n=$ odd

$J\Big|_{(0,n\pi)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\lambda = \pm i$

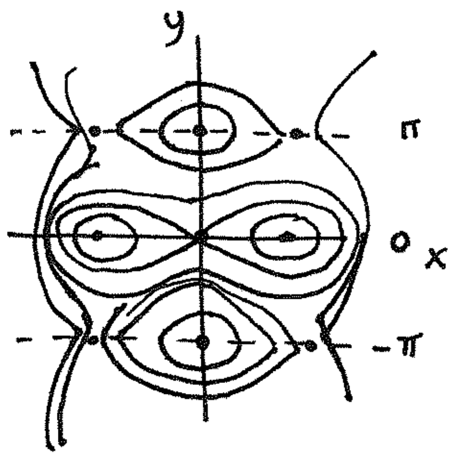
centers.

$J\Big|_{(1,n\pi)} = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}$

$\lambda = \pm 2$

saddle nodes.

Jacobin for $(-1, n\pi)$ same $\nearrow$



The same picture will repeat.

7. $\dot{x} = y + x - x^3$

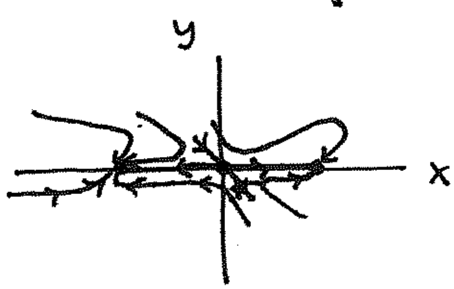
$\dot{y} = -y$

three fixed points $y = 0$, $x = 0, 1, -1$ $\quad (0,0)\ (-1,0)\ (1,0)$

$$J = \begin{bmatrix} 1-3x^2 & 1 \\ 0 & -1 \end{bmatrix}$$

$J\big|_{(0,0)} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ $\quad \lambda = \pm 1$ saddle point.

$\lambda_1 = 1$ $V_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \lambda_2 = -1$ $V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



$J\big|_{(-1,0)} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$ $\quad \lambda = -1, -2$ stable node.

$J\big|_{(1,0)} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$ $\quad \lambda = -1, -2$ stable node.

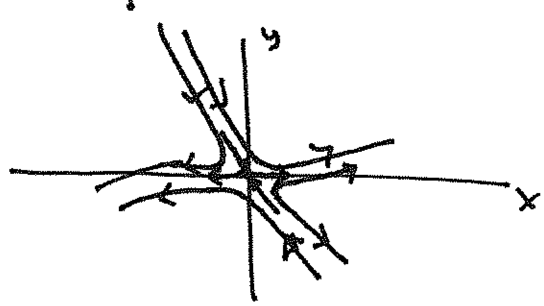
8. $\dot{x} = 1 + y - e^{-x}$

$\dot{y} = x^3 - y$

one fixed point at $(0,0)$

$J = \begin{bmatrix} e^{-x} & 1 \\ 3x^2 & -1 \end{bmatrix}$ $\quad J\big|_{(0,0)} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ $\quad \lambda = \pm 1$ Saddle point.

$\lambda_1 = 1$ $V_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \lambda_2 = -1$ $V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



Short Answer - Essay

9. Explain in words what each of the following things are, how you find (or draw) them and what they tell you about a nonlinear system. Give an example of each.

One Dimensional Flow on a Line

Fixed Point

Potential Function for 1-D system

Bifurcation Diagram

Phase Portrait (2-D)

Saddle-Node Bifurcation

Transcritical Bifurcation

Pitchfork Bifurcation (sub and supercritical)

Stable and Unstable Manifold

Euler Method, Improved Euler Method, and Runge Kutta

see class notes!