Homework 4 – Solutions.

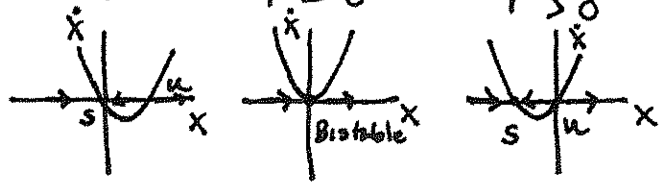
Bifurcations.

1. $\dot{x} = rx + x^2$

Fixed Points: $rx + x^2 = 0$ $\quad x(r+x) = 0$

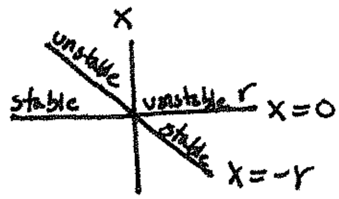
$x^* = 0$ and $x^* = -r$ TWO FP's

cases $\quad r < 0 \quad r = 0 \quad r > 0$

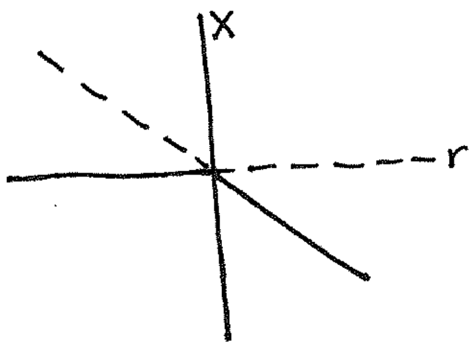


FLOW ON LINE.
Desmos – online.

DIAGRAM
plot $x=0$
and $x=-r$



SKETCH



BIFURCATION
DIAGRAM

$r=0$ is the bifurcation point.

transcritical bifurcation.

2. $\dot{x} = x + x^2 + r$

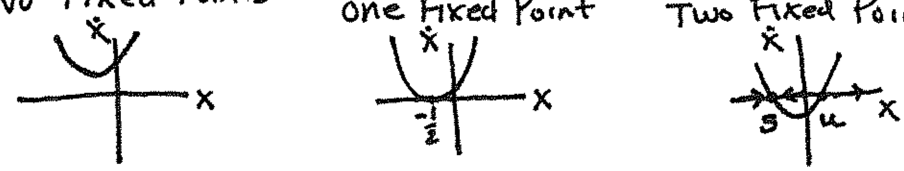
Fixed Points $\quad x^2 + x + r = 0$ quadratic eqn.

$$x^* = \frac{-1 \pm \sqrt{1-4r}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4r}$$

cases: $\quad 1-4r < 0 \quad\quad 1-4r = 0 \quad\quad 1-4r > 0$

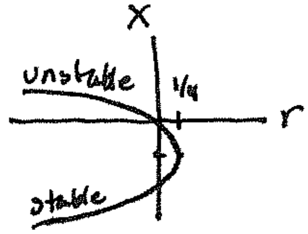
$\frac{1}{4} < r \quad\quad r = \frac{1}{4} \quad\quad \frac{1}{4} > r$

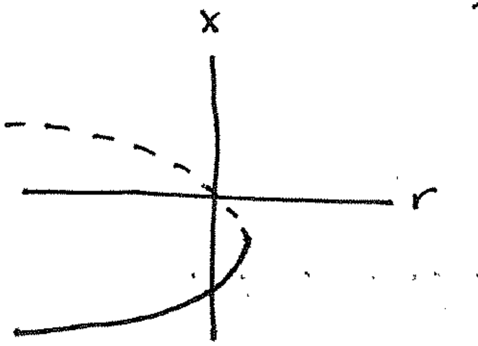
No Fixed Points $\quad$ One Fixed Point $\quad$ Two Fixed Points.



Desmos – online.

Diagram
plot $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4r}$



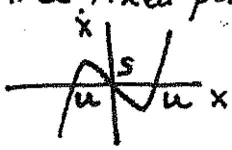


BIFURCATION
DIAGRAM
$r = \frac{1}{4}$ bifurcation point
Saddle node bifurcation

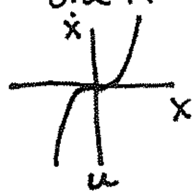
3. $\dot{x} = rx + x^3$
Fixed Points: $rx + x^3 = 0$ $\quad x(r + x^2) = 0$
$x = 0 \quad x = \pm\sqrt{-r}$

cases

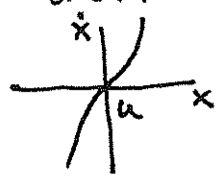
$r < 0$
Three fixed points



$r = 0$
one FP

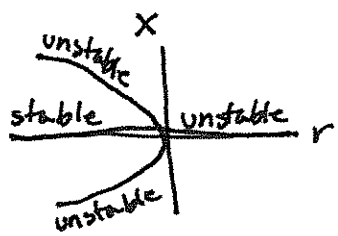


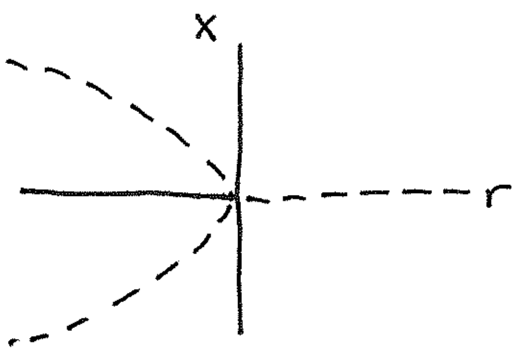
$r > 0$
one FP



Desmos – online

Diagram.
plot $x = 0 \quad x = \pm\sqrt{-r}$





BIFURCATION
DIAGRAM
$r = 0$ bifurcation point
Subcritical Pitchfork.

4. $\dot{x} = x + rx^2 + x^3$

Fixed points: $x(1 + rx + x^2) = 0$

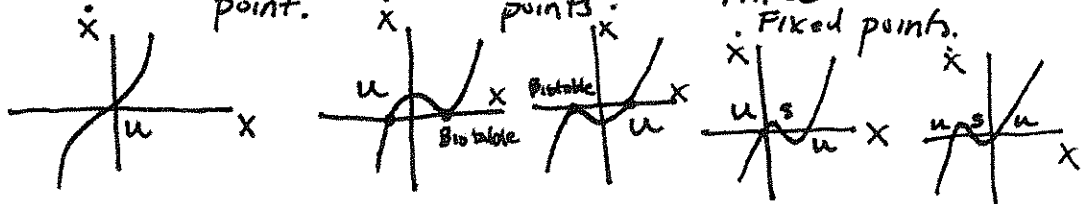
$x = 0 \qquad x^2 + rx + 1 = 0$

$$x = \frac{-r \pm \sqrt{r^2 - 4}}{2}$$

Cases: $r^2 - 4 < 0 \qquad r^2 - 4 = 0 \qquad r^2 - 4 > 0$

$-2 < r < \pm 2 \qquad r = \pm 2 \qquad r < -2 \quad r > 2$

one fixed point. Two fixed points. Three Fixed points.

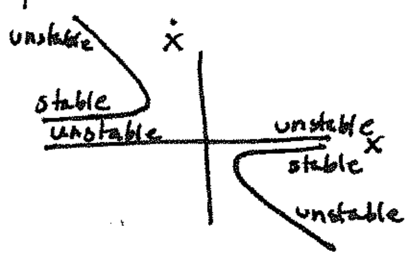


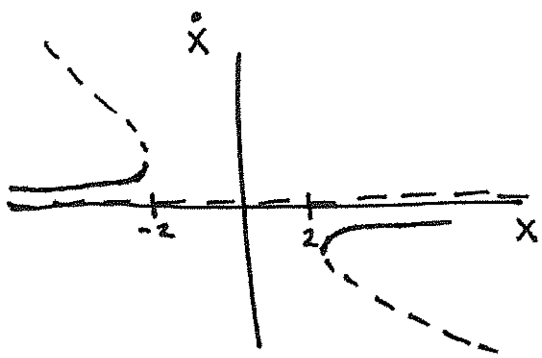
$r = -2 \qquad r = 2 \qquad r < -2 \qquad r > -2$

Desmos online.

diagram:

plot $x = 0$ and $x = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - 4}$





Bifurcation Diagram

$r = \pm 2$ Bifurcation points.

This looks like a saddle node bifurcation.

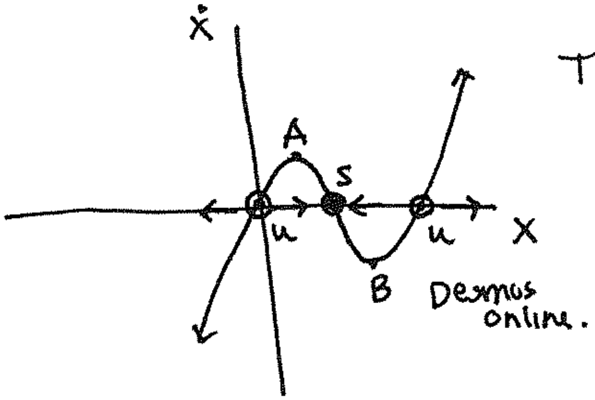
First Order System

For the following equation, find the fixed points, analyze the stability of the fixed points first using **flow on a line** then using **linearization**.

Next, discuss what happens to your solution: Where is the magnitude of the velocity the fastest? What happens as you start from different initial conditions? Are there any existence or uniqueness problems?

$$\dot{x} = x^3 - 3x^2 + 2x$$

Flow on a Line



Three fixed points.

$x=0$ unstable
$x=1$ stable
$x=2$ unstable

} Values from looking at graph.

Linearization $\quad x^3 - 3x^2 + 2x = 0 \qquad x=0$

$x(x^2-3x+2)=0 \qquad x=1 \qquad$ algebraically.

$x(x-1)(x-2)=0 \qquad x=2$

$f'(x) = 3x^2 - 6x + 2$

$f'(0) = 2 > 0$ unstable

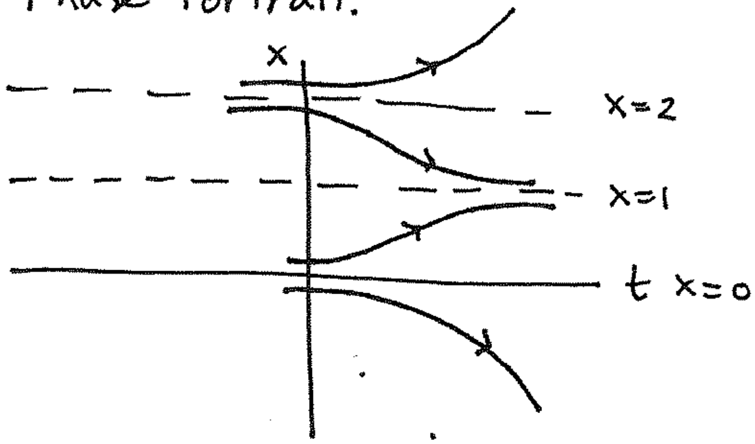
$f'(1) = 3-6+2 = -1 < 0$ stable

$f'(2) = 12-12+2 = 2 > 0$ unstable.

Velocity is fastest where $x \to \pm\infty$
this is where the magnitude of $\dot{x}$ is largest.

locally the velocity is fastest at points A and B

Phase Portrait.



If we start between
$0 < x < 2$
our soln $\to 1$ as $t \to \infty$
otherwise the soln $\to \pm\infty$

There are no existence of uniqueness issues
because

$$\left.\begin{array}{l} f(x) = x^3 - 3x^2 + 2x \\ \text{and} \\ f'(x) = 3x^2 - 6x + 2 \end{array}\right\}$$

are continuous and bounded for all reasonable initial conditions

Second Order System

We are going to do a full analysis for an arms race model. This should use most of the techniques that we have learned for second order systems.

1. First read the full article *Modifying the Richarsdon Arms Race Model with a Carrying Capacity.*
Use the parameter values given in the paper and analyze the nonlinear system (1b) and (2b)

Get **as much information as you can** about the system. This could include fixed points, linearization about fixed points, categorizing the fixed points, finding nullclines, sketching the phase portrait, and plotting the phase portrait using PPLANE. Just do your absolute BEST!

Now that you have a full analysis of each model, **write a one page, typed, conclusion for what the model predicts**. What are the different possibilities for the arms race, depending on how many arms each country initially has? Are there any best case scenarios? Worst case scenarios?

Finally, have some fun with this model. Investigate what happens as you change the parameters in the nonlinear model. For this analysis, just use PPLANE and interpret the results. Can you choose the parameters in such a way that you totally change the system? Can you change the stability, location, or existence of fixed points? What changes do you seen in your solution for new parameters? What do these new parameters mean in terms of the arms race? Do this for at least on interesting case, **print out the PPLANE plot and write next to the plot any interesting features you see and what happens to the solution.**

Second Order System

$$\dot{x} = \left(1 - \frac{x}{K_1}\right)(ay - mx + r)$$

$$\dot{y} = \left(1 - \frac{y}{K_2}\right)(bx - ny + s)$$

Find fixed points

$$(1 - x/K_1)(ay - mx + r) = 0$$

$$x = K_1 \quad \text{or} \quad x = \frac{1}{m}(ay + r)$$

$$(1 - y/K_2)(bx - ny + s) = 0$$

$$y = K_2 \quad \text{or} \quad y = \frac{1}{n}(bx + s)$$

Cases:

these are the same

$$x = K_1 \quad \text{and} \quad y = K_2$$

$$x = K_1 \quad \text{and} \quad y = \frac{1}{n}(bK_1 + s)$$

$$y = K_2 \quad \text{and} \quad x = K_1$$

$$y = K_2 \quad \text{and} \quad x = \frac{1}{m}(aK_2 + r)$$

$$x = \frac{1}{m}(ay + r) \quad \text{and} \quad y = \frac{1}{n}(bx + s)$$

$$x = \frac{1}{m}\left(a\,\frac{abr + sm}{nm - ab} + r\right)$$

$$= \frac{1}{m}\left(\frac{a^2br + asm + rnm - rab}{nm - ab}\right)$$

$$y = \frac{1}{n}\left(\frac{b}{m}(ay + r) + s\right)$$

$$= \frac{ab}{nm}y + \frac{1}{n}\left(\frac{abr}{m} + s\right)$$

$$\left(1 - \frac{ab}{nm}\right)y = \frac{abr}{nm} + \frac{s}{n}$$

$$y = \frac{\frac{abr + sm}{nm}}{\frac{nm - ab}{nm}} = \frac{abr + sm}{nm - ab}$$

Four Fixed Points:

$$(k_1, k_2) \quad \left(k_1, \frac{1}{n}(bk_1+s)\right) \left(\frac{1}{m}(ak_2+r), k_2\right)$$

$$\left(\frac{1}{m}\left(\frac{a^2br+asm}{nm-ab}+r\right), \frac{abr+sm}{nm-ab}\right)$$

These are a little messy... but lets look at the Jacobian.

$$J = \begin{bmatrix} -\frac{1}{k_1}(ay-mx+r) + \left(1-\frac{x}{k_1}\right)(-m) & \left(1-\frac{x}{k_1}\right)a \\ (1-y/k_2)(b) & -\frac{1}{k_2}(bx-ny+s)+(1-y/k_2)(-n) \end{bmatrix}$$

Test (k_1, k_2)

$$J\Big|_{(k_1 k_2)} = \begin{bmatrix} -1/k_1(ak_2-mk_1+r) & 0 \\ 0 & -1/k_2(bk_1-nk_2+s) \end{bmatrix}$$

$$\tau = -\left(a\frac{k_2}{k_1} - m + \frac{r}{k_1}\right) - \left(\frac{bk_1}{k_2} - n + \frac{s}{k_2}\right)$$

$$= m+n - \frac{r+ak_2}{k_1} - \frac{s+bk_1}{k_2}$$

This is getting messy — use PPLANE.

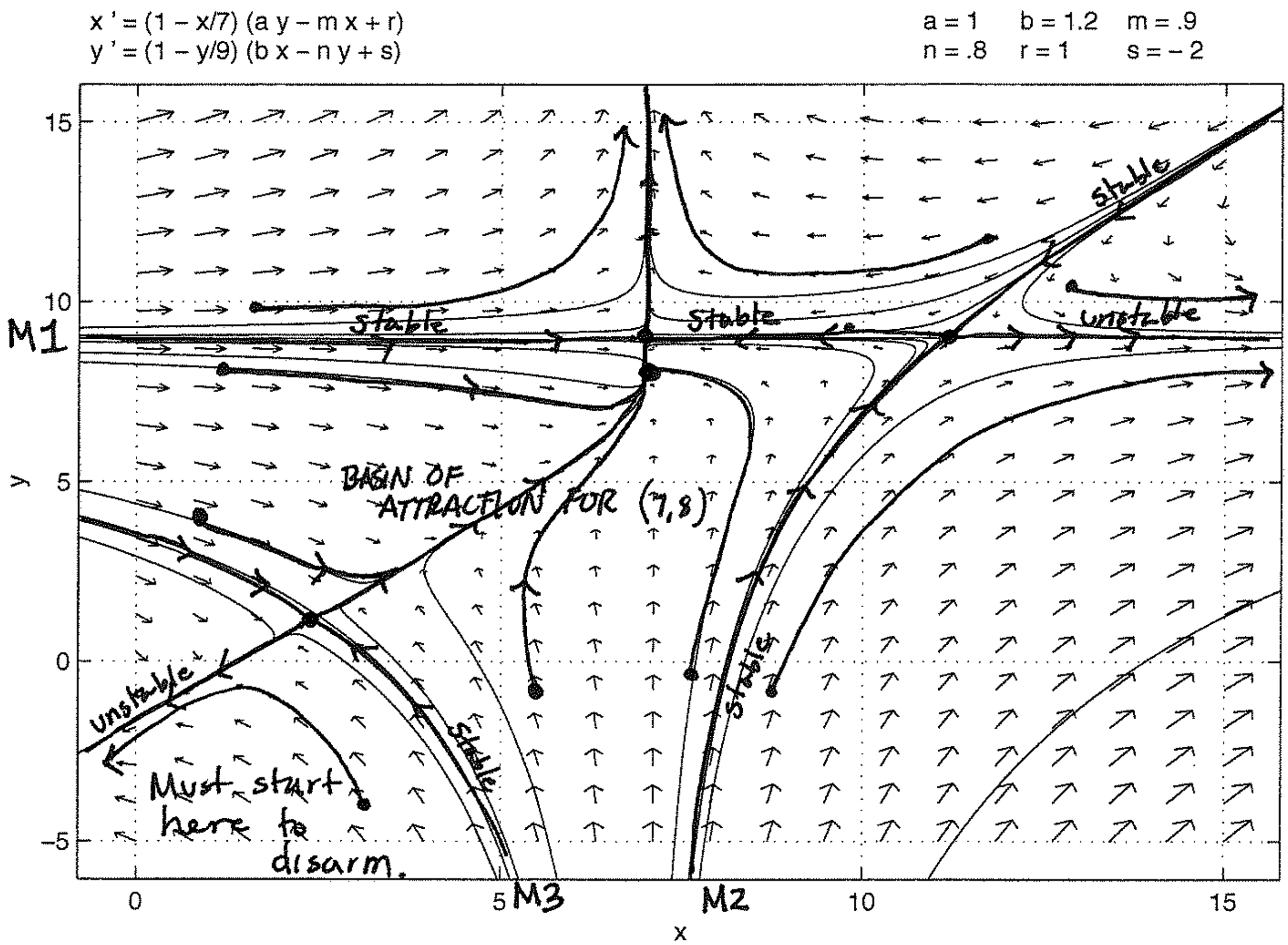
Fixed points $(k_1, k_2) = (7, 9)$ saddle

$\left(k_1, \frac{1}{n}(bk_1 + s)\right) = (7, 8)$ stable — both countries spend on arms

$\left(\frac{1}{m}(ak_2 + r), k_2\right) = (11.111, 9)$ saddle

$\left(\frac{1}{m}\left(\frac{a^2br + asm}{nm - ab} + r\right), \frac{abr + sm}{nm - ab}\right) = (2.5, 1.25)$ saddle.

solved on Matlab.



The Manifolds divide phase space!

The Stable Manifolds — going toward a saddle — are specifically important!

Start above M1 ~~xxx~~ $y \to \infty$ and x follows manifold

below M1 go to stable node at (7,8)

left of M2

right of M2 ~~x~~ $x \to \infty$ and y follows manifold.

above M3 go to stable node at (7,8)

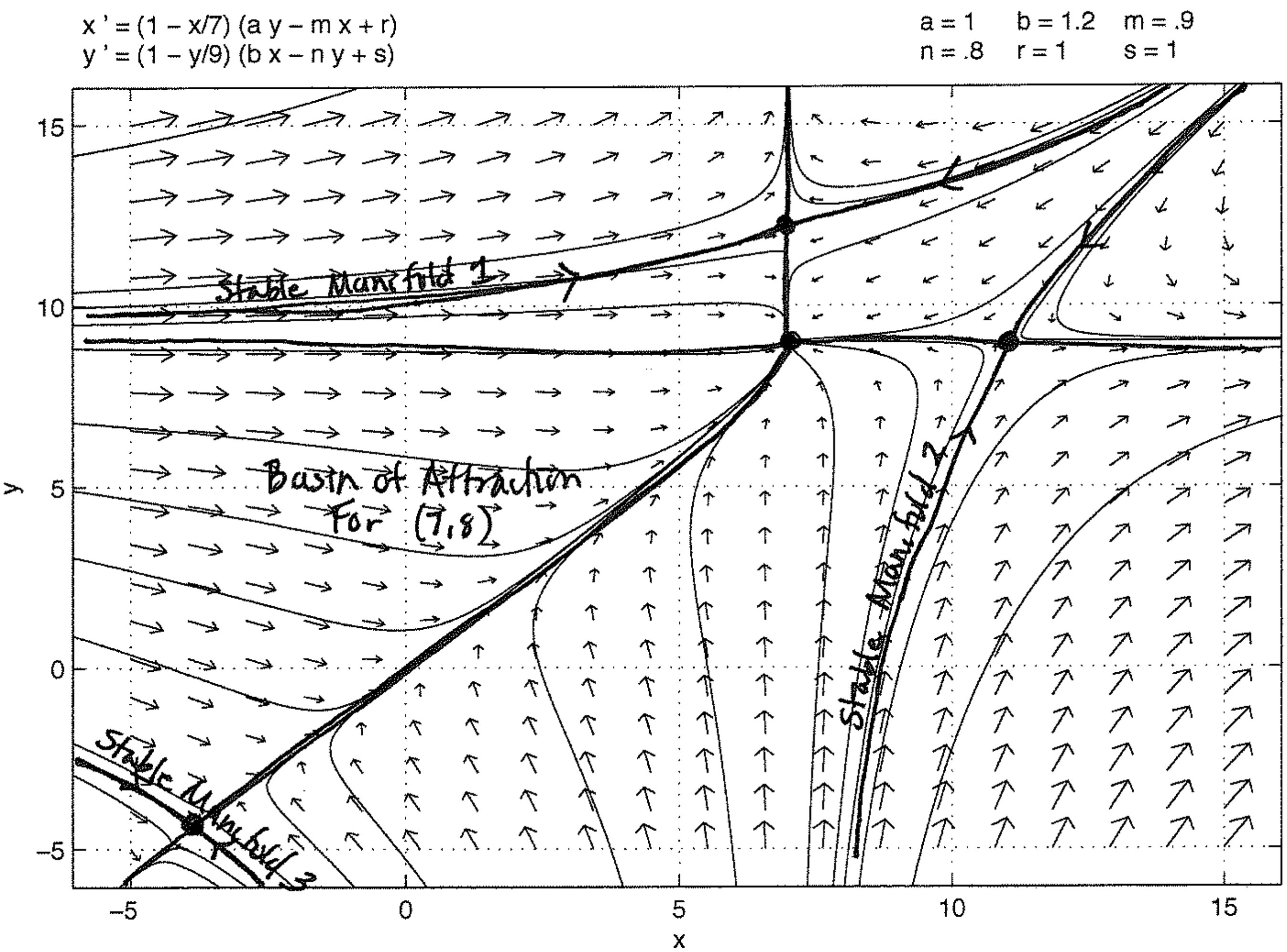
below M3 both $x, y \to -\infty$

Change $s=1$ Countries equally hostile

FP: $(7,9)(7,11.75)(11.11,9)(-3.75,-4.375)$

stable — saddle. — Solved on Matlab.

$s=1$ They have same hostility value.



This expands the basin of attraction for the point (7,9) – now both countries go to carrying capacity.

This point means that both countries are spending on arms but it is not expanding or getting worse.

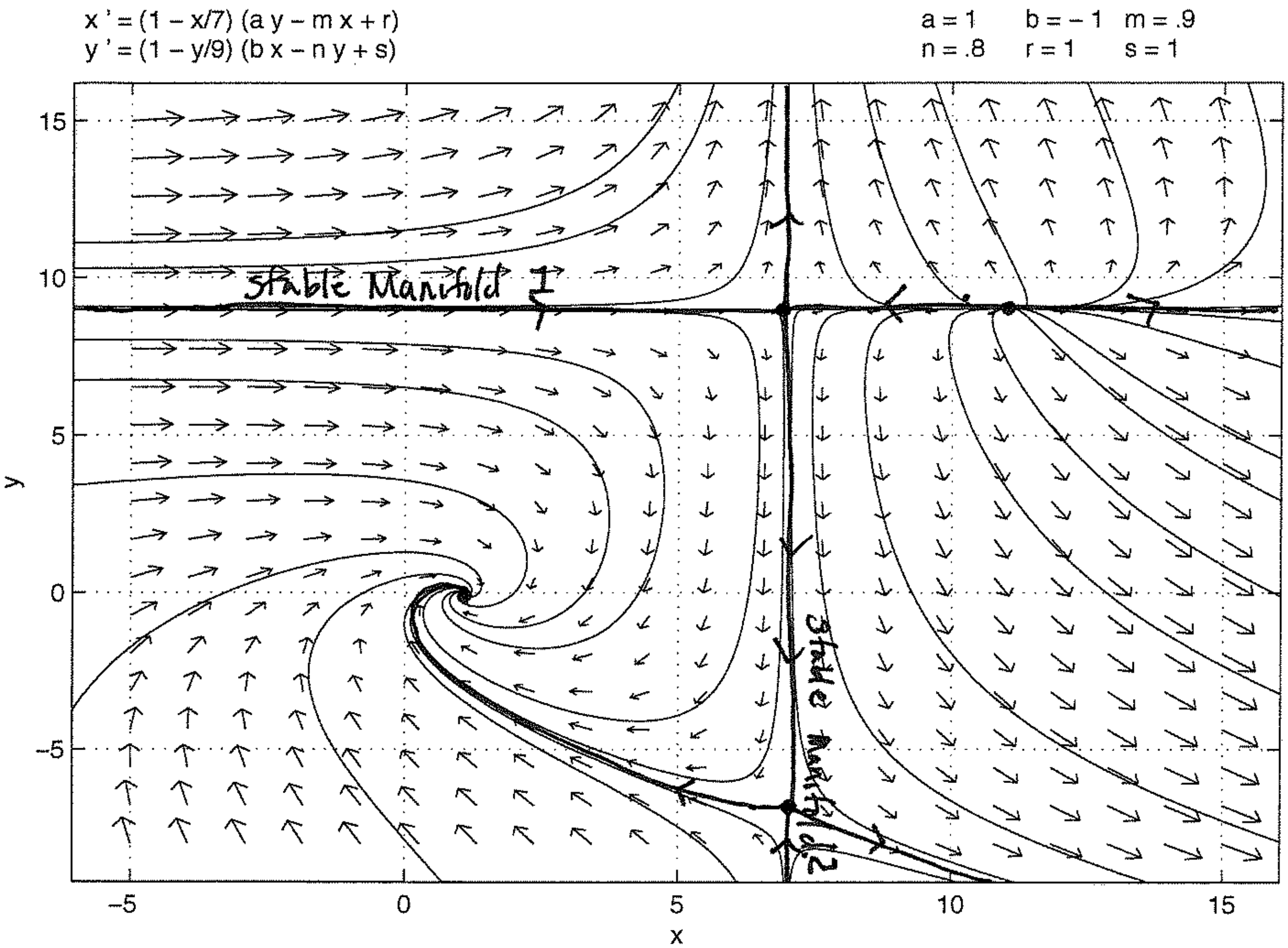
Also we see it take more to "disarm" must start below Stable manifold 3 for any hope of disarming.

saddle saddle unstable node stable spiral.

Fixed Points: $(7,9)$ $(7,-7.5)$ $(11.111,9)$ $(1.0465,-.0581)$

solved on Matlab.

b = -1
and s = 1 have the same hostility value but y has an opposite reaction to x's behavior.



Now if the countries start in the bottom left they are attracted to the stable spiral fixed point.

Above Stable Manifold 1 we would have an arms race with y - increasing rapidly

Below and to the right of Manifold 2 we would have y-selling and x-buying.