

Nonlinear Dynamics and Chaos – Week 1 Homework

Professor:

Dr. Joanna Bieri

joanna_bieri@redlands.edu

DAY 1

Reduction to a First Order System

1. $y'' + 6y' + 9y = 0$

2. $y^{(4)} - 8y'' + 16y = 0$

3. $y''' + 2y' - y = \cos(t)$

Separable First Order ODE's

4. $\dot{y} + 2y = 0$

5. $\dot{y} = \frac{1}{\sin y}$

6. $\dot{y} = ty^3$

7. $y^3 \dot{y} = (y^4 + 1) \cos t$

DAY 2

Flow's on the Line

1. For the following exercise consider the ODE

$$\dot{x} = \sin(x)$$

- Find all the fixed points of the flow
- At which points x does the flow have the greatest velocity to the right?
- Find the flow's acceleration $\ddot{x}$ as a function of x .
- Find the points where the flow has maximum positive acceleration.

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch typical solution curves for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

2. $\dot{x} = 4x^2 - 16$

3. $\dot{x} = 1 - x^{14}$

4. $\dot{x} = x - x^3$

5. $\dot{x} = e^{-x} \sin(x)$

6. $\dot{x} = 1 + \frac{1}{2} \cos(x)$

7. $\dot{x} = 1 - 2 \cos(x)$

8. $\dot{x} = e^x - \cos(x)$

Hint: Sketch the graphs of e^x and $\cos(x)$ on the same axes, and look for intersections. You won't be able to find the fixed points explicitly, but you can still discuss the qualitative behavior.

9. (Working backwards, from flows to equations)

Given the equation $\dot{x} = f(x)$, we know how to sketch the corresponding flow on the real line. Now we are going to solve the opposite problem:

- Draw a phase portrait for a system that has a bistable fixed point at $x = -1$, a stable fixed point at $x = 0$ and an unstable fixed point at $x = 2$.
- For the phase portrait that you just drew, find a first order differential equation that it could have come from. (There is more than one correct answer for this problem.)

10. For each of the following, find an equation $\dot{x} = f(x)$, with the stated properties, or if there are no examples, explain why not. (In all cases, assume that $f(x)$ is a smooth function.)

- Every real number is a fixed point.
- Every integer is a fixed point.
- There are precisely three fixed points, and all of them are stable.
- There are no fixed points.
- There are precisely 100 fixed points.

11. (Analytical solution for charging capacitor)

Obtain the analytical solution of the initial value problem, with

$$Q(0) = 0$$

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$$

Population Growth

12. (Exact Solution to the Logistic Equation)

There are two ways to solve the logistic equation

$$\dot{N} = rN \left(1 - \frac{N}{K} \right)$$

We will assume the arbitrary initial condition $N(0) = N_0$.

a) Use the separable method, as learned in class, and integrate using partial fractions.

b) Make the change of variables $x = \frac{1}{N}$. First take the derivative of x then solve for $\dot{N}$ which should appear in your expression for x thanks to the chain rule. Also, note that $N = \frac{1}{x}$. Substitute into the logistic equation for N and $\dot{N}$ leaving a much simpler ODE, in terms of x . Solve this new ODE and substitute back to get in terms of N .

13. (Tumor growth)

The growth of cancerous tumors can be modeled by the Gompertz law:

$$\dot{N} = -aN \ln(bN)$$

where $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters.

a) Interpret a and b biologically. What do they mean? Take time to think about this and be imaginative in your solution.

b) Sketch the vector field on the real line and then graph typical solution curves for $N(t)$ and various initial values. What does your graphical solution tell you? The predictions of this simple model agree surprisingly well with data on tumor growth, as long as N is not too small.

DAY 3

Linear Stability Analysis

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, then use a graphical argument to decide the stability:

1. $\dot{x} = x(1 - x)$

2. $\dot{x} = x(1 - x)(2 - x)$

3. $\dot{x} = \tan x$

4. $\dot{x} = x^2(6 - x)$

5. $\dot{x} = 1 - e^{-x^2}$

6. $\dot{x} = \ln(x)$

7. $\dot{x} = ax - x^3$ where a can be positive negative or zero. Discuss all three cases. Are they different or the same and how?

Existence and Uniqueness

8. Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is a fixed constant. Show that $x(t) \rightarrow \pm\infty$ in finite time, starting from any initial condition other than $x(0) = 0$.

DAY 4

Potentials

For each of the following vector fields, plot the potential function $V(x)$ and identify all the equilibrium points and their stability.

1. $\dot{x} = x(1 - x)$

2. $\dot{x} = 3$

3. $\dot{x} = \sin(x)$

4. $\dot{x} = 2 + \sin(x)$

5. $\dot{x} = -\sinh(x)$ where $\sinh(x)$ is the hyperbolic sine function. Remember, you can always look things up in a book or online if they don't look familiar!

6. $\dot{x} = r + x - x^3$ for various values of r . What happens as you change r ?

Numerical Methods

Now consider the initial value problem

$$\dot{x} = x + e^{-1}, \quad x(0) = 0$$

In contrast to many other problems we have solved, it does not have an analytical solution. This means we must use a numerical method to solve the problem and we cannot exactly predict the error. In the code you will not be able to compare to an exact solution. Please do the following:

1. Use **dfield** to plot a slope field and some typical solution curves. Answer the following questions: Does this equation have fixed points? Do you see any fixed points on your slope field? Save a copy of the slope field graph in your Google folder.
2. On your hand written homework - give a qualitative description of the solution. This should include a Linear Stability Analysis.
3. Using **Freemat or Matlab** Solve the equation numerically. You should edit the code we used in class. How small does your time step need to be? (Remember that if your time step is small enough then your solutions will converge and decreasing the time step

even more will not make a big change in the solution.) Save a copy of your code as `euler.m` in your Google folder.

4. Repeat problem 2 using the Improved Euler method. Save a copy of your code as `improvedEuler.m` in your Google folder.
5. Repeat problem 2 using the Runge-Kutta method. Save a copy of your code as `rungeKutta.m` in your Google folder.

For each of these problems you should write up some information in your handed in homework. Talk about how you discretized the equation for each method and discuss what you found from your numerical solutions.