Nonlinear Dynamics and Chaos - Week 2 Homework

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DAY 1

Saddle-Node Bifurcations

For each of the following problems:

- Sketch all of the qualitatively different vector fields that occur as r is varied. (flow on the line)
- Show that a saddle-node bifurcation occurs at a critical value of r, and determine that value. (graphically)
- Do a linear stability analysis and show mathematically when the bifurcation occurs. (fixed points and stability using derivative - when the problem is sufficiently simple!)
- Finally, sketch the bifurcation diagram of fixed points x^* versus r. (MOST IMPORTANT - draw this correctly - it should match what you found above!)

1. $\dot{x} = 1 + rx + x^2$
2. $\dot{x} = r - \cosh x$
3. $\dot{x} = r + x - ln(1+x)$
4. $\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$

DAY 2

Transcritical Bifurcations

For each of the following problems:

- Sketch all of the qualitatively different vector fields that occur as r is varied. (flow on the line)
- Show that a transcritical bifurcation occurs at a critical value of r, and determine that value. (graphically)
- Do a linear stability analysis and show mathematically when the bifurcation occurs. (fixed points and stability using derivative - when the problem is sufficiently simple!)
- Finally, sketch the bifurcation diagram of fixed points x^* versus r. (MOST IMPORTANT - draw this correctly - it should match what you found above!)

1. $\dot{x} = rx + x^2$
2. $\dot{x} = rx - ln(1+x)$
3. $\dot{x} = x - rx(1-x)$
4. $\dot{x} = x(r - e^x)$

Taylor Series

Expand the following functions around the indicated point, n, using Taylor series. Write your final approximation by neglecting, or ignoring, $O(x^3)$ terms.

1. $f(x) = e^{-x}, \quad n = 5$
2. $f(x) = sin(x), \quad n = 4$
3. $f(x) = \frac{1}{1-x}, \quad n = 4$
4. $f(x) = \sqrt{1+x}, \quad n = 3$

Pitchfork Bifurcations

For each of the following problems:

- Sketch all of the qualitatively different vector fields that occur as r is varied. (flow on the line)
- Show that a pitchfork bifurcation occurs at a critical value of r, and determine that value. (graphically)
- Do a linear stability analysis and show mathematically when the bifurcation occurs. (fixed points and stability using derivative - when the problem is sufficiently simple!)
- Finally, sketch the bifurcation diagram of fixed points x^* versus r. (MOST IMPORTANT - draw this correctly - it should match what you found above!)

1. $\dot{x} = rx + 4x^3$
2. $\dot{x} = rx - \sinh x$
3. $\dot{x} = rx - 4x^3$
4. $\dot{x} = x + \frac{rx}{1+x^2}$

DAY 3

General Bifurcations

The next exercises are designed to test your ability to distinguish among the various types of bifurcations - it is easy to confuse them so this might be a challenge. In each case: For each of the following problems:

- Sketch all of the qualitatively different vector fields that occur as r is varied. (flow on the line)
- Determine what kind of bifurcation occurs and determine the critical value of r. (graphically)
- Do a linear stability analysis and show mathematically when the bifurcation occurs. (fixed points and stability using derivative - when the problem is sufficiently simple!)
- Finally, sketch the bifurcation diagram of fixed points x^* versus r. (MOST IMPORTANT - draw this correctly - it should match what you found above!)

1. $\dot{x} = r - 3x^2$
2. $\dot{x} = 5 - re^{-x^2}$
3. $\dot{x} = rx - \frac{x}{1+x^2}$
4. $\dot{x} = rx - \frac{x}{1+x}$

DAY 4

Nondimensionalization and Analysis

1. Consider the system modeled by

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) + \frac{BN^2}{A^2 + N^2}$$

This system has too many parameters to analyze easily. Use the nondimensioanization from class

$$x = \frac{N}{A}$$

$$\tau = \frac{B}{A}t$$

to simplify the system. Show all your work and follow the examples from the book and in class to make sure you understand how to do the substitution.

2. Figuring out what the nondimensional variables should be can be difficult. This problem walks you through that process. Consider the ODE

$$\frac{dx}{dt} = a + bx + cx^2$$

where the coefficients a, b and c are all constants that must necessarily have different units (Can you figure out why they have different units?). To simplify this equation we will define the general nondimensionalization as

$$\xi = \frac{x}{x_0}$$

$$\tau = \frac{t}{t_0}$$

where we want ξ and τ to be unitless (nondimensionalized). Then our goal is to figure out what x_0 and t_0 need to be to simplify the problem as much as possible. Remember that nondimensionalization is not unique so two people can easily come up with different nondimensional parameters and equations.

To do this we take the following steps:

- Using ξ and τ, find derivatives and plug them into the ode as if you were doing a normal nondimensionalization.
- After simplifying you should find

$$\frac{d\xi}{d\tau} = \frac{t_0 a}{x_0} + (t_0 b)\xi + (t_0 x_0 c)\xi^2$$

- Now choose x_0 and t_0 to make the right hand side as simple as possible (HELP [1])
- Once you have defined both of your nondimensional variables you should be left with just one term full of constants, define this as your only remaining parameter α.

This should leave you with

$$\frac{d\xi}{d\tau} = 1 + \xi + \alpha\xi^2$$

Now do the following

1. Analyze the system using the techniques from class (Flow on Line, Phase Portrait, Bifurcation Diagram, Fixed Points ,Stability). Consider the cases of $\alpha < 0$, $\alpha = 0$, $0 < \alpha < \frac{1}{4}$, $\alpha = \frac{1}{4}$, and $\alpha > \frac{1}{4}$ separately. Why are these the four cases?
2. Assume that x has the units of energy (Joule) and t has units of time (second). What are the units of a, b, and c? What are the units of α?
3. Now instead imagine that x measures the amount of activism or extremism for a political or religious group over time t measured in weeks as they interact via social media and read the news. If x<0 but not too large this is like not really caring much about world events, if $x > 0$ but not too large, then this is like being politically involved but not extreme, if $x \to -\infty$ this is not caring at all and becoming a hermit, and if $x \to \infty$ this is dangerous extremism. In this (somewhat contrived) case
   - a would represent some external forcing - group members reading posts about world events or political or economic pressure

[1] HINT 1: if $t_0 = \frac{1}{b}$ then the term in front of ξ becomes one. HINT 2: if $x_0 = at_0$ then the first, constant, term becomes 1

from outside groups. This is the general tone of the news the group reads. If $a < 0$ then this means that the group seeks news that does not match it's viewpoint.

- b represents an average internal reaction - in the absence of social interactions and reading the news, group members would still reflect on their feelings and react. If $b = 0$ then this is like brain washing, individuals no longer listen to self but only to the group and external forcing.
- c represents the impact of interactions within a group - reading posts from people within the group who share your viewpoint. This is the general activism or extremism of the group as they interact. If $c < 0$ then the group tries to calm or reverse it's overall temperament.

We can see α as being the ratio between the impact of external forcing, the news and the social group, (ca) and the impact of self reflection or individual brain power (b^2). What does your analysis of the system tell you about the ratio of these two things? Talk about each of the regions of the solution in terms of this nonlinaer analysis.