

Nonlinear Dynamics and Chaos – Week 4 Homework

Professor:

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DAY 1

For the following examples: Solve for the nullclines and plot them, then talk about what kind of bifurcation you see and what the critical parameter value for μ appears to be. Then find the fixed points and analyze the system using the Jacobian and eigenvalues for the difference cases for μ - do the best you can. Plot some phase planes for which you expect qualitatively different solutions as μ varies.

1. $\dot{x} = \mu x - x^2$ $\dot{y} = -y$
2. $\dot{x} = \mu x + x^3$ $\dot{y} = -y$
3. $\dot{x} = \mu x - x^3$ $\dot{y} = -y$
(shortcut for 3 - use what you found in number 2)
4. $\dot{x} = y - 2x$ $\dot{y} = \mu + x^2 - y$

3. Challenge problem: Consider the following equations which models a laser in two variables:

$$\begin{aligned}\dot{n} &= GnN - kn \\ \dot{N} &= -GnN - fN + p\end{aligned}$$

where N is the number of excited atoms, n is the number of photons, and G, K, f , and p are parameters. Only p may take on negative values. For the physics folks: G is the gain coefficient, k is the decay rate due to loss of photons, f is the decay rate for spontaneous emission, and p is the pump strength. Milonni and Eberly (1988)

For this laser system do the following:

1. Nondimensionalize the system: let $x = \frac{n}{n_0}$, $y = \frac{N}{N_0}$ and $\tau = \frac{t}{t_0}$ then sub in and try to set parameter combinations to simplify the equation. You should be able to get:

$$\begin{aligned}\frac{dx}{d\tau} &= x(y - 1) \\ \frac{dy}{d\tau} &= -xy - ay + b\end{aligned}$$

as your new system with parameters $a = \frac{f}{k}$ and $b = \frac{pG}{k^2}$

2. Find and classify all fixed points in this system - they will be in terms of your parameters.
3. Sketch the qualitatively different phase planes that occur and the parameters are varied.
4. What types of bifurcation can occur?

DAY 2 and DAY 3

Start reviewing for the final exam - we will just cover fun and interesting topics in class - exploring chaos or fractals. We can also work on review problems and present solutions in class.

Nonlinear Dynamics and Chaos

Practice Final Exam Problems

The exam is open book and open notes. You will be allowed to use computers (calculators) to help with graphing or any calculations, reference books from the library, or reference articles. You will **not** be allowed to work with other humans, students or professors (in this class or not). Your work must be your own. If you take information or graphs from a computer or a book, just say where you got your information, a small note in the margins is good enough. Your work needs to be neat and readable, this is your chance to show what you have learned in this class. **Write each problem neatly on a separate page.**

For this homework - try to do as much as possible on your own and then check your answers with your classmates. This will allow you to practice for the actual exam.

First Order Systems

For the following equations, find the fixed points, analyze the stability of the fixed points first using flow on a line then using linear stability analysis. Then find the potential and compare the stability results, do all the different analyses match up?

Finally, discuss what happens to your solution: Where is the magnitude of the velocity the fastest? What happens as you start from different initial conditions? Draw the phase portrait (typical solution trajectories).

$$\dot{x} = x^3 - 3x^2 + 2x$$

$$\dot{x} = x^2(6 - x)$$

Second Order Systems

Find the fixed points and analyze their stability for the following systems. This should include nullclines, eigenvalues, and eigenvectors. Then sketch the phase plane and clearly explain how you can see the fixed, nullclines, eigenvalues, and eigenvectors (for real valued vectors) in the phase plane.

$$\dot{x} = y$$

$$\dot{y} = -2x - 3y$$

$$\dot{x} = -3x + 4y$$

$$\dot{y} = -2x + 3y$$

$$\dot{x} = x - y$$

$$\dot{y} = x^2 - 4$$

Bifurcations

For the following first order problems, **do a full bifurcation analysis for the parameter r** . This should include a bifurcation diagram with stability lines, analysis of the fixed points for different cases of r , a description in words of what kind of bifurcation it is, and a description in words of what would happen to your solution starting from different initial conditions.

1. $\dot{x} = rx + x^2$

2. $\dot{x} = x + x^2 + r$

3. $\dot{x} = rx + x^3$

4. $\dot{x} = x + rx^2 + x^3$

Numerical Solutions

CHOOSE ONE of the problems above and write some numerical code to solve the problem. You may use any numerical method you want. Please check that your solution makes sense based on the parameter choice and initial conditions that you choose (compare with your diagrams).

HINTS

You can use DFIELD to check your solution to your first order system or PPLANE for second order systems

Wolfram alpha, Simbolab, Desmos and Freemat/Matlab are available to help you plot your solutions, use them as much as you want, just write that you used them in your solution. The classroom and classroom computers should be available.

Other topics

The following topics are fair game for the final exam:

- Fixed points, flow on a line, potentials, and linear stability analysis
- First order systems - bifurcations.
- Imperfect parameters and bifurcations.
- Numerical methods for first order equations.
- Phase portraits and bifurcation diagrams
- Reducing a second order ODE to a first order system of ODEs.
- Two dimensional systems (linear and nonlinear)
- Phase Planes and Nullclines
- Eigenvalues, Eigenvectors, and the Jacobian.
- Analyzing a "real world" problem using nonlinear dynamics analysis.