
Numerical Analysis

FINAL EXAM - Fall 2023

Please be very clear in your write up. Explain what you are solving, what methods you are using, why your solutions make sense. Please do your own work, **NO TEAM WORK!** This should reflect your own individual thinking and process and any suspect work is grounds for failing the class.

Part 1 - General Questions - No Code Required

You may submit a .pdf (latex or scan of work) or a Jupyter Notebook for this section.

- **Do three out of the seven problems.** Be very clear in your explanations of your work. Answers with no explanations will not be graded.
 - **Write each problem up in a separate section of markdown or on a separate piece of paper.** Please clearly label which problem you are doing and keep your solutions in order so it is easy to grade.
 - Please write neatly and clearly explain your calculations. It is your responsibility to communicate your understanding, not your professors job to fill the gaps in your logical process and understanding.
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G1. Given the points $(1, \frac{1}{2}), (2, \frac{1}{4}), (3, \frac{1}{6})$, note that these are points on the curve $f(x) = \frac{1}{2x}$, construct the Lagrange interpolating polynomial $P_2(x)$. Bound the error in this polynomial, where we note that the nodes are evenly spaced. Explain what you are doing at each step.

G2. Write Psuedo code that describes the steps in the Bisection Method for finding the root to a polynomial. In other words how does the Bisection Method work? Why does it make sense for rootfinding? For what cases does it converge?

G3. Write down the Taylor series expansion for $f(x) = e^x$ about the point $a = 1$. How many terms would be required in the series approximation to estimate $f(2)$ with an accuracy of 10^{-5} ? Use this expansion to integrate

$$\int_0^1 e^{x^2} dx$$

G4. Describe how Newtons Method works to find the root of a polynomial. Use words and a graph to show how we derived this. In other words, given an initial guess x_0 discuss how we find x_1 . What are some issues that could arise that would cause Newtons Method to diverge or converge to the wrong root? (It's okay to draw pictures to help explain your answer.)

G5. Consider the following iterative method:

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{5}{x_n} \right)$$

What are the fixed points for this system? Discuss whether or not this iterative method will converge to the **positive fixed point** on some range $[a, b]$.

G6. Consider the derivative:

$$f''(x) \frac{2f(x) - 5f(x+h) + 4f(x+2h) - f(x+3h)}{h^2}$$

Show using Taylor series expansions about $f(x)$ that this really does approximate the second derivative. What is the error in this method?

G7. Show how we derived the Trapezoidal Rule for numerical integration starting from a linear interpolating polynomial and give some pseudo code for how you would implement it. Discuss how we derived Richardson's Extrapolation. How is this used to improve our results from the Trapezoidal Rule?

Part 2 - Coding Questions

You must submit a Jupyter Notebook for this section.

- **Do one out of the three problems.** Be very clear in your explanations of your work. Answers with no explanations will not be graded.
 - **Write clearly commented easy to follow code.** Please clearly label which problem you are doing and describe your work using markdown so it is easy to grade.
 - Please write neatly and clearly explain your calculations. It is your responsibility to communicate your understanding, not your professors job to fill the gaps in your logical process and understanding.
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C1. The differential equation

$$\frac{dh}{dt} = -0.6\pi r^2 \sqrt{-2g} \frac{\sqrt{h}}{A(h)}$$

where $A(h) = \pi h^2$, models liquid flowing from an inverted conical tank with a circular hole in the bottom. In this equation r is the radius of the hole, h is the height of the liquid in the tank, measured from the bottom (tip of the cone), and $A(h)$ is the area of the cross section of the tank h -units above the hole. Suppose $r = 0.2 \text{ ft}$, $g = -32.17 \frac{\text{ft}}{\text{s}^2}$ and that initially the water level $h = h_0$.

1. Draw a sketch of the problem.
2. Are there any values of h_0 for which the differential equation might have existence or uniqueness problems? Explain why this might make physical sense.
3. Choose a numerical method to solve this problem. State what method you are using and explain how it was derived and the error associated with the method.
4. Do you expect any issues with the convergence of your method?
5. Use your numerical method to compute the water level after 10 seconds if $h_0 = 4$.
6. Determine approximately, when the tank will be empty.

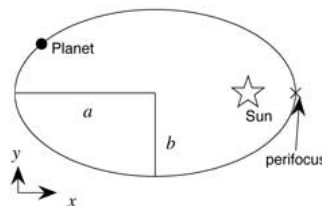
In your write-up please include the extremely well commented code that you wrote using the verbatim command. You should make sure to clearly answer the questions above.

C2. Consider the following interpolation problem: Find values for a , b , c , and d , for $y = ax^3 + bx^2 + cx + d$ to interpolate a cubic polynomial assuming you are given four points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , (x_4, y_4) . We are going to use matrix inverse methods to solve for these points rather than our polynomial interpolation formulas.

1. Write down a system of four equations (one for each of the points (x_i, y_i)) for four unknowns (a, b, c, d) .
2. State this problem as a matrix problem of the form $\mathbf{A}\mathbf{x} = \mathbf{b}$.
3. Discuss the plusses and minuses of using a direct or iterative method on this system of equations, this should include a discussion of convergence for the iterative methods. Which method should you choose?
4. Write code that uses the chosen numerical method (not matrix inverse) to solve the system of equations for a, b, c , and d . State which method you are using and why.
5. Finally, using the numbers generated from your numerical solution, plot the cubic along with the points to demonstrate the interpolation is working.
6. Test your code using simple values. For example points that you know will give a straight line. Then use the points $(1, 2.7)$, $(1.5, 3)$, $(2, 3.5)$, $(2.5, 4)$.

In your write-up please include the extremely well commented code that you wrote using the verbatim command. You should make sure to clearly answer the questions above. Also write down the interpolating polynomial and show a graph of the function for the points given. How would you generalize this code to interpolate n -points?

C3.



A planet's movement around the sun can be expressed parametrically, at time t , by the equations:

$$x(t) = a(\cos(E(t)) - e)$$

$$y(t) = a\sqrt{1 - e^2} \sin(E(t))$$

according to Kepler's Laws of planetary motion. Here a is the semi-major axis (for Earth $a = 152,098,232$), e is the eccentricity of the planets orbit (for Earth $e = 0.0167$), and E is the eccentric anomaly given by

$$E(t) = \omega t + e \sin(E(t))$$

where ω is the frequency of orbit, in radians, (for Earth this is $\omega = \frac{2\pi}{365.25635}$). We have assumed units for time in days and length in kilometers. We assume that when $t = 0$ the planet is located at the perifocus, where the planet is closest to the sun. Notice that we cannot solve directly for E with an exact formula. This is one of the most famous examples of a nonlinear root finding equation in science. We can however use a root finding method to solve for E at a specific time, t , and then plug values into $(x(t), y(t))$ to find the planets location and plot the orbit around the sun. We are going to write code that solves for and plots the planets location.

1. First write the equation for E as a root finding problem, $f(E) = 0$.
2. Now choose a root finding method to solve this problem. Discuss why you chose that method. What possible issues might you have?
3. Write code that will solve for the earths location at any time t , using your method to find the root $f(E) = 0$ and then plugging in to $x(t)$ and $y(t)$. Here you will use a for loop though time to calculate the position of the earth for time step. You can use an error tolerance of 10^{-3} for this problem.
4. Then plot $x(t)$ vs. $y(t)$ to see the parametric curve of the earths orbit.

In your write-up please include the extremely well commented code that you wrote using the verbatim command. You should make sure to clearly answer the questions above. You should also include a picture of the planetary orbit and a table of values that gives t , $x(t)$ and $y(t)$ at the following times: $t = 0, 100, 200$.