Numerical Analysis
Day 12.

Start talking about Polynomial Interpolation

GOAL – given a set of data points
find a formula (Polynomial)
whose graph passes through
those points.

When doing an experiment ⟶ TAKE DATA
what does that data look like?
$(t_0, x_0)\ (t_1, x_1)\ (t_2, x_2) \ldots$

Usually we want to do something with this:

A. Predict what will happen in the future t_{100}
B. Fill in the gaps what happened between t_1 and t_2
C. Develop a theory ... come up with an equation that matches the data.

This is also useful in making graphics more pleasing to the eye — smooth curves.

START WITH A SET OF DATA POINTS ...

these can be a wide range of functions.

given a set of functions $p_i(x)$ we look for

$$p(x) = a_0 + a_1 p_1(x) + a_2 p_2(x) + \cdots + a_n p_n(x)$$

We find the coefficients $a_0, a_1, \ldots, a_n$ so that $p(x)$ goes through our points.

START BY ASSUMING P_i are POLYNOMIALS.

In Mathematics — START SIMPLE and BUILD on WHAT YOU KNOW.

<u>LINEAR INTERPOLATION</u> – if I have only two data points to start, the best I can do is linear.

$$P_1(x) = a_0 + a_1 x.$$

Given two points (x_0, y_0) and (x_1, y_1) find the straight line passing through these points.

$$y_0 = mx_0 + b \longrightarrow b = y_0 - mx_0$$

$$y_1 = mx_1 + b \longrightarrow y_1 = mx_1 + y_0 - mx_0$$

solve for m

$$y_1 - y_0 = m(x_1 - x_0) \qquad m = \frac{y_1 - y_0}{x_1 - x_0}$$

so $y_0 = \dfrac{y_1 - y_0}{x_1 - x_0} x_0 + b$ solve for b

$$b = y_0 - \frac{y_1 x_0 - y_0 x_0}{x_1 - x_0} = \frac{y_0 x_1 - \cancel{y_0 x_0} - y_1 x_0 + \cancel{y_0 x_0}}{x_1 - x_0}$$

$$b = \frac{y_0 x_1 - y_1 x_0}{x_1 - x_0}$$

$$P_1 = \frac{y_1 - y_0}{x_1 - x_0} x + \frac{y_0 x_1 - y_1 x_0}{x_1 - x_0} = \frac{x_1 - x}{x_1 - x_0} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

REWRITE
gather y_0 y_1

$$P_1(x) = L_0(x) y_0 + L_1(x) y_1$$

L_0, L_1 are Lagrange Basis Functions for our Linear Interpolation.

<u>Ex</u> I have the points $(0,0)$ and $(1,1)$

$$P_1 = \frac{1-x}{1-0}(0) + \frac{x-0}{1-0} 1 = x$$

$P_1(x) = x$ is the line that fits these points!

As expected!

QUADRATIC INTERPOLATION — If we have three data points to start then we can define a second order polynomial.

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

given (x_0, y_0) (x_1, y_1) and (x_2, y_2)

we write $P_2(x) = L_0(x) y_0 + L_1(x) y_1 + L_2(x) y_2$ Lagrange Formula.

where
$$\left.\begin{aligned} L_0 &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \\ L_1 &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\ L_2 &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \end{aligned}\right\} \text{Lagrange Basis Functions.}$$

A property of these basis functions:

$$L_i(x_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} = \delta_{ij} \quad \text{Kronecher Delta}$$

For quadratic interpolation $L_i(x)$ are all degree 2. So $P_2(x)$ must have degree ≤ 2.

QUESTION Are these polynomials unique?
$P_1(x)$ — straight line ... this is pretty obvious.

Lets show UNIQUENESS for the Lagrange formula for $P_2(x)$.

Assume that there is another polynomial, $Q(x)$ such that

$$\deg(Q) \leq 2 \quad \text{and} \quad Q(x_i) = y_i \quad i = 0,1,2$$

There is another degree 2 polynomial that perfectly fits our points.

let $R(x) = P_2(x) - Q(x)$ ... then
$$\deg(R(x)) \leq 2$$
$$\text{and} \quad R(x_i) = P_2(x_i) - Q(x_i) = y_i - y_i = 0 \quad i=0,1,2$$

This means that $R(x)$ has THREE distinct roots.
BUT $\deg(R(x)) \leq 2$ can't have three roots so $R(x) = 0$.
and $P_2(x) = Q(x)$ □

GENERALIZED TO HIGHER DEGREE.

Given the data points $(x_0, y_0)(x_1, y_1) \ldots (x_n, y_n)$ construct a polynomial $P_n(x)$, $\deg\{P_n\} \leq n$ such that $P_n(x_i) = y_i \quad i = 0, 1, 2, \ldots$

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x)$$

where $L_i(x)$ are the Lagrange Basis Functions:

$$L_i(x) = \frac{(x-x_0)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

$i = 0, 1, 2 \ldots$

$$L_i(x_k) = \delta_{i,k}.$$

WRITE PSUEDO CODE ... WORKSHEET:

Read in: $(0,-1)(1,-1)(2,7)$ $\quad X_{vals} = [0, 1, 2]$
$\quad Y_{vals} = [-1, -1, 7]$

Construct $L_i(x)$:

Need $X = 0 : .1 : 2$ values for graphing the interpolating polynomial

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-2)}{2}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-2)}{-1}$$

For $i = 1 : \text{length}(x)$ PROBLEM.
$L(i,:) = (x - X_{val}(i)) .* (x - X_{val}(i+1))$

HARD TO GENERALIZE.

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{2}$$

WOULD NEED TO JUST TYPE THESE IN!

Construct $P_2(x)$:

$$P_2(x) = L_0 y_0 + L_1 y_1 + L_2 y_2$$
$$= \frac{(x-1)(x-2)}{2}[-1] + \frac{x(x-2)}{-1}[-1] + \frac{x(x-1)}{2}[7]$$

GRAPH

plot (x, P_2).

- The Lagrange formula is good if you are doing the calculations by hand... BUT it is hard to program and generalize!!

There is another way to calculate these polynomials:

NEWTONS DIVIDED DIFFERENCE INTERPOLATION FORMULA.

Define: The divided difference $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

- discrete form of the derivative .. this is the FIRST ORDER divided difference.

Then HIGHER ORDER divided differences are defined using the lower ones.

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$ SECOND ORDER.

- similar to saying the second derivative is the derivative of the first derivative.

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$ THIRD ORDER.

IN GENERAL:

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots x_n] - f[x_0, x_1, \ldots x_{n-1}]}{x_n - x_0}$$

This is related to the derivative by the MEAN VALUE THEOREM.

$$f[x_0, x_1, \ldots, x_n] \doteq \frac{1}{n!} f^{(n)}(c)$$

NOTE: We can permute the x_i with no effect to the divided difference

$$f[x_0, x_1] = f[x_1, x_0]$$

To use this in interpolation:

Should feel similar to Taylor Polynomial.

$$P_1(x) = f(x_0) + (x-x_0)f[x_0,x_1]$$
$$P_2(x) = f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2]$$
$$\vdots$$
$$P_n(x) = f(x_0) + (x-x_0)f[x_0,x_1] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f[x_0,x_1,\ldots,x_n]$$

Now we can construct our polynomials term by term just by adding the next divided difference.

Notation $D^k f(x_i) = f[x_i, x_{i+1}, \ldots, x_{i+k}]$

WRITE PSUEDO CODE ... WORKSHEET:

<u>Read in</u> $(0,-1)(1,-1)(2,7)$ $\quad$ Xvals $= [0,1,2]$ $\quad$ Yvals $= [-1,-1,7]$

<u>Construct the Divided Differences.</u> need first and second since I have THREE POINTS

$$f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{(-1)-(-1)}{1-0} = 0$$

SNEAKY WAY TO DO THIS!!

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{8-0}{2} = 4$$

NEED $\quad f[x_1,x_2] = \dfrac{f(x_2)-f(x_1)}{x_2-x_1} = \dfrac{7-(-1)}{1} = 8$

need $x=0:.1:2$ x values to evaluate the polynomial.

<u>Construct the Polynomial.</u>

$$\begin{aligned} P_2(x) &= f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] \\ &= -1 + (x-0)[0] + (x-0)(x-1)[4] \\ &= -1 + 4x(x-1) \end{aligned}$$

GRAPH

NEWTON

$$\begin{aligned} P_2(x) &= 4x(x-1) - 1 \\ &= 4x^2 - 4x - 1 \end{aligned}$$

LAGRANGE

$$\begin{aligned} P_2(x) &= -\frac{(x-1)(x-2)}{2} + x(x-2) + \frac{7}{2}(x-1)x \\ &= -\tfrac{1}{2}[x^2-3x+2] + x^2-2x + \tfrac{7}{2}[x^2-x] \\ &= \left[1+\tfrac{7}{2}-\tfrac{1}{2}\right]x^2 + \left[\tfrac{-7}{2}-2+\tfrac{3}{2}\right]x + [-1] \\ &= 4x^2-4x-1 \quad \checkmark \quad \text{SAME} \end{aligned}$$

SNEAKY WAY TO COMPUTE: DIVIDED DIFFERENCE

Read in the Xvals and Yvals: $Xvals = [x_0, x_1, \dots]$
$Yvals = [f(x_0), f(x_1), \dots]$

<u>Set</u> DIVDIFF = Yvals

$DIVDIFF = [f(x_0), f(x_1), \dots f(x_n)]$

Loop forward $i = 2$: lenght(DIVDIFF)
Loop backward j = length(DIVDIFF) : -1 : i

Calculate the Divided Difference
$$DIVDIFF(j) = \frac{DIVDIFF(j) - DIVDIFF(j-1)}{Xval(j) - Xval(j-i+1)}$$

end
end.

<u>From the example</u>: $Xvals = [0, 1, 2]$ $Yvals = [-1, -1, 7]$

$D = [-1, -1, 7]$
for $i = 2:3$
for $j = 3:-1:i$

$\underline{i=2}$, $\underline{j=3}$ $D(3) = \dfrac{D(3) - D(2)}{Xval(3) - Xval(2)} = \dfrac{7-(-1)}{2-1} = 8$

$\underline{j=2=i}$ $D(2) = \dfrac{D(2) - D(1)}{Xval(2) - Xval(1)} = \dfrac{-1-(-1)}{1-0} = 0$

$D = [-1, 0, 8]$

$\underline{i=3}$, $\underline{j=3=i}$

$D(3) = \dfrac{D(3) - D(2)}{Xval(3) - Xval(1)} = \dfrac{8-0}{2-1} = 4$

$D = [-1, 0, 4]$

Then to construct the polynomial:

$$P_2 = D(1) + (x - Xval(1))D(2) + (x - Xval(1))(x - Xval(2))D(3)$$
$$= -1 + (x-0)[0] + (x-0)(x-1)[4] \quad \checkmark$$