Numerical Analysis
Day 11

Last Time Polynomial Interpolation.

Given a set of points (x_i, y_i) we seek a polynomial that fits those points.

$P_n(x)$ such that $\deg\{P_n\} \le n$ and
$P_n(x_i) = y_i \quad i = 1, 2, \ldots, n.$

Lagrange Formula:
How did we derive this? Linear, Quadratic...

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x)$$

where L_k are the Lagrange Basis Functions.

$$L_k(x) = \prod_{i \neq k} \frac{x - x_i}{x_k - x_i} = \frac{(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Problem: Harder to Program for a general case.

Newtons Divided Difference

$$P_1(x) = f(x_0) + (x-x_0) f[x_0, x_1]$$
$$P_2(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2]$$
$$\vdots$$
$$P_n(x) = P_{n-1}(x) + (x-x_0)(x-x_1)\ldots(x-x_{n-1}) f[x_0, \ldots, x_n]$$

This is computationally easier! Why?

$$P_n(x) = D_0 + (x-x_0)D_1 + (x-x_0)(x-x_1)D_2 + \cdots + (x-x_0)\ldots(x-x_{n-1})D_n$$
$$= D_0 + (x-x_0)\Big[D_1 + (x-x_1)\Big[D_2 + (x-x_2)\Big[D_3 + \ldots \; (x-x_{n-1})D_n\Big]\Big]\ldots\Big]$$

USE NESTED MULTIPLICATION !!

NOTE: Remember that these $P_n(x)$ are unique... how did we show this?

Ex Use Polynomial Interpolation to approximate $y = \cos(x)$ on the range $[1,2]$ with an N-degree polynomial.

Need to choose Xvals.

EVENLY SPACED:

$$dx = \frac{(b-a)}{N-1};$$

$$Xvals = a : dx : b;$$

NOT EVENLY SPACED

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for k=1:N
    Xval(k) = f(k);
end
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Must choose a function that maps 1:N to the range $[a, b]$

RANDOMLY SPACED

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for k=1:N
    Xval(k) =
end.
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Then $Yvals = \cos(Xvals)$

evaluate the function the "Hard Way" to get these points.

USE YOUR INTERPOLATION CODE TO GET $P_n(x)$

USE $P_n(x)$ to evaluate the approximate points.

Easily evaluate the approximate points

Example: interpeval(0, 4, 5, 100, 0)

- $\underbrace{0, 4}_{[a,b]}$
- 5 → order
- 100 → N = # of approximation points
- 0 → 0 = EVEN, 1 = NOT EVEN, 2 = RANDOM.

Notice the difference in error!

Error in Polynomial Interpolation.

$$\text{Error} = f(x) - P_n(x)$$

but this is cheating because we would need to calculate all the $f(x)$ to get the error.

Remember: The goal is to find $P_n(x)$ without knowing $f(x)$ in general.

Assume we are interpolating $f(x)$ at the points x_i $i = 1, 2 \ldots n$ then

$$P_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

Lagrange Formulation.

assuming $f(x)$ has $n+1$ continuous derivatives on $[a, b]$ then the error is

This should look similar to the remainder term from Taylor Polynomials

$$f(x) - P_n(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!} f^{(n+1)}(c_x)$$

What Theorem

for $a \le x \le b$ and c_x between the max and min values for x_i.

We use this to bound the error.

Ex Consider the function $y = \ln(x)$. Assume we are approximating this on $[1, 10]$ with 2 points x_0, x_1 where $1 \le x_0 < x_1 \le 2$

What degree Polynomial?

$$P_1(x) = y_0 L_0(x) + y_1 L_1(x) \quad (\text{degree} \le 1)$$

$$E(x) \equiv \text{Error} = f(x) - P_1(x) = \frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c_x)$$

$$f'(x) = \frac{1}{x} \qquad f''(x) = \frac{-1}{x^2}$$

$$E(x) \equiv \text{Error} = \frac{(x-x_0)(x-x_1)}{2}\left(\frac{-1}{c_x^2}\right)$$

Let's BOUND THE ERROR!

How bad could it be?

bring in the negative

$$|E(x)| \le \left|\frac{(x-x_0)(x_1-x)}{2}\right|\left(\frac{1}{x_0^2}\right) \le \frac{1}{2}|(x-x_0)(x_1-x)| = \frac{1}{2}|\psi(x)|$$

the biggest this could be is at x_0 since $1 \le x_0 < x_1 \le 10$

The error depends on our choice of x_0 and x_1 along with the location of the approximation.
Note – what is $E(x_0)$ and $E(x_1)$? Does this make sense?

consider $\psi(x) = (x-x_0)(x_1-x)$

lets maximize this function.
FROM CALC ... How do we find a maximum?

$$\psi'(x) = \frac{d}{dx}(x-x_0)(x_1-x) = \frac{d}{dx}\left[xx_1 - x_0x_1 - x^2 + x_0x\right]$$

$$= x_1 - 2x + x_0 = 0$$

solve for x_{max}: $x_{max} = \frac{x_0+x_1}{2}$
How are we sure this is max?

$$\max\{\psi(x)\} = \psi\{x_{max})$$

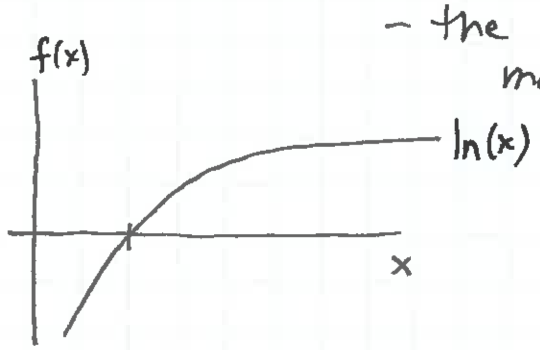
$$= \left(\frac{x_0+x_1}{2} - x_0\right)\left(x_1 - \frac{x_0+x_1}{2}\right) = \left(\frac{x_1-x_0}{2}\right)\left(\frac{x_1-x_0}{2}\right) = \frac{h^2}{4}$$

where $h = x_1 - x_0$

So $|E(x)| \le \frac{1}{2}|\psi(x)| \le \frac{h^2}{8}$ • Use absolute value because $(x-x_0)$ or (x_1-x) could be negative.

implications – the error depends on the distance between x_0 and x_1

– the error is greater near $x=1$ ... this makes sense ... $\ln(x)$ is "curvy" here



YOU TRY Consider $y=e^x$ on $[0,1]$. Interpolate with two points $0 \le x_0, x_1 \le 1$. Bound the error.

$$E(x) = \frac{(x-x_0)(x-x_1)}{2!} f^{(2)}(c_x) = \frac{1}{2}(x-x_0)(x-x_1)e^{c_x}.$$

$$|E(x)| \le \frac{1}{2}\left|(x-x_0)(x-x_1)\right| e \le \frac{e}{2}\frac{h^2}{4} = \frac{eh^2}{8}$$

where $h = x_1 - x_0$

- The error depends on choice of x_1 and x_0
- It will be better on the $x=0$ side.

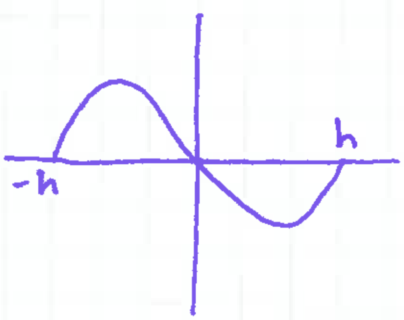
CHALLENGE What is the error if we used THREE POINTS $0 \le x_0, x_1, x_2 \le 1$ $y=e^x$?

ONLY IN NOTES.

$$E(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} f^{(3)}(c_x) = \frac{1}{6}(x-x_0)(x-x_1)(x-x_2)e^{c_x}$$

$$|E(x)| \le \frac{1}{6}\left|(x-x_0)(x-x_1)(x-x_2)\right| e$$

Need to Maximize $\psi(x) = (x-x_0)(x-x_1)(x-x_2)$



TRICK... the largest this would be is evenly spaced:

$$\frac{(x-h)(x-0)(x+h)}{6}$$

$$w(x) = \frac{1}{6}x^3 - \frac{xh^2}{6}$$

$$w'(x) = \frac{1}{2}x^2 - \frac{h^2}{6} = 0 \qquad x^2 = \frac{h^2}{3} \qquad x = \pm\frac{h}{\sqrt{3}}$$

$$w(\pm h/\sqrt{3}) = \frac{h^3}{9\sqrt{3}}$$

$$\boxed{|E(x)| \le \frac{h^3}{9\sqrt{3}} e}$$