Numerics Day 17

[redacted]

Today we start Numerical Integration.

Goal: Evaluate $I = \int_a^b f(x)\,dx$

this is a definite integral so the soln we seek is a number.

Fundamental Theorem: If $F(x)$ is an antiderivative of $f(x)$ then

$$I = F(x)\Big|_a^b = F(b) - F(a)$$

The problem is that it is often messy or impossible to find $F(x)$.

★ If you cannot solve a problem then replace it with a "near-by" problem that you can solve.

$$I \approx \int_a^b \tilde{f}(x)\,dx \equiv \tilde{I}$$

many numerical schemes are based on finding a good approximate $f(x)$.

ERROR $\quad E = I - \tilde{I} = \int_a^b [f(x) - \tilde{f}(x)]\,dx$

$$|E| \le \int_a^b |f(x) - \tilde{f}(x)|\,dx \le (b-a) \max_{a \le x \le b} |f(x) - \tilde{f}(x)|$$

How do we choose approximate $\tilde{f}(x)$?

1. Taylor polynomials approximating $f(x)$
2. Polynomial Interpolation.

<u>Ex</u>

$$I = \int_0^1 e^{x^2}\,dx$$ use a Taylor Polynomial to approximate e^{x^2}

1. Construct the Taylor Polynomial (including the error term)

$$e^t = 1 + t + \frac{1}{2}t^2 + \cdots + \frac{1}{n!}t^n + \frac{1}{(n+1)!}t^{n+1}e^{c_t}$$

let $t = x^2$

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \cdots + \frac{1}{n!}x^{2n} + \frac{1}{(n+1)!}x^{2n+2}e^{d_x}$$

where $0 \le d_x \le x^2$

2. Integrate the Taylor Polynomial.

$$I = \int_0^1 1 + x^2 + \frac{1}{2}x^4 + \cdots + \frac{1}{n!}x^{2n}\,dx + \frac{1}{(n+1)!}\int_0^1 x^{2n+2}e^{d_x}\,dx$$

$$= \left[x + \frac{1}{3}x^3 + \frac{1}{2}\cdot\frac{1}{5}x^5 + \cdots + \frac{1}{n!}\frac{1}{2n+1}x^{2n+1}\right]\Big|_0^1 + \frac{1}{(n+1)!}\frac{1}{2n+3}x^{2n+3}e^{d_x}\Big|_0^1$$

3. Choose n and bound the error.

$n=3$ then $$I = \left[x + \frac{1}{3}x^3 + \frac{1}{2}\cdot\frac{1}{5}x^5 + \frac{1}{3!}\frac{1}{7}x^7\right]\Big|_0^1 + E$$

$$= 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} + E = 1.4571 + E.$$

where $$0 < E \le \frac{e^{d_x}}{4!}\frac{1}{9}x^9\Big|_0^1 \le \frac{e}{24}\cdot\frac{1}{9} = \frac{e}{216} = .0126$$

This is a simple idea ... but difficult to do numerically
– a lot to do by hand here ...

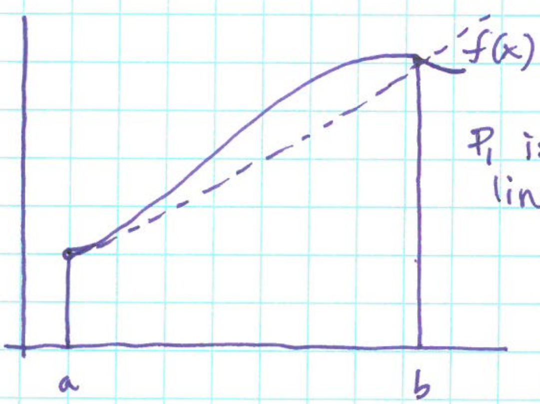
If a computer is doing the "dirty" work then it is easier to construct polynomial interpolates.

In otherwords ... approximate $f(x)$ using interpolation then integrate or add them up.

Start by assuming that we have evenly spaced nodes! This can all be generalized to non-evenly spaced nodes.

LINEAR INTERPOLATION.

$$P_1(x) = \frac{(b-x)f(a) + (x-a)f(b)}{b-a}$$



P_1 is a straight line between a and b interpolating $f(x)$.

$$\int_a^b P_1(x)\,dx = \frac{1}{b-a}\Big[f(a)\big[bx - \tfrac{1}{2}x^2\big] + f(b)\big[\tfrac{1}{2}x^2 - ax\big]\Big]_a^b$$
$$= \frac{1}{2}\frac{(b-a)^2}{(b-a)}\big[f(a)+f(b)\big]$$

Then $I = \int_a^b f(x)\,dx = \frac{1}{2}(b-a)\big[f(a)+f(b)\big]$

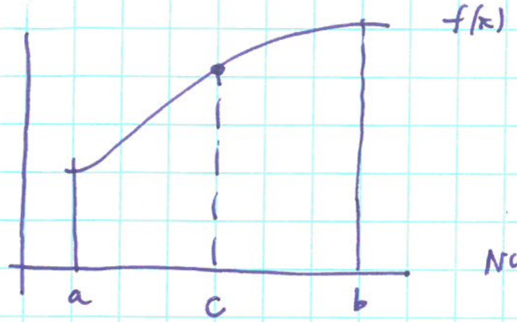
we could also see this by area under the curve

$$I \approx T_1(f) \equiv \frac{1}{2}(b-a)\big[f(a)+f(b)\big]$$ TRAPEZOIDAL RULE.

This could contain a LOT of error... How do we make it more accurate?

TWO WAYS

1. Increase the degree of approximation to better match the curve
2. Consider Piecewise Linear Polynomial Interpolation.



Then

$c = \frac{b+a}{2}$ $\qquad I = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

NOTE. here we choose equal sub intervals but this is not required.

Then $I = \frac{c-a}{2}\left[f(a)+f(c)\right] + \frac{b-c}{2}\left[f(c)+f(b)\right]$

$$= \frac{h}{2}\left[f(a) + 2f(c) + f(b)\right] \equiv T_2(f) \qquad h = \frac{b-a}{2}$$

If we continue doing this n-times with $h = \frac{b-a}{n}$

then we get x-nodes: $x_j = a + jh \quad j = 0, 1, \ldots, n$

AND $I \approx h\left[\frac{1}{2}f(a) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(b)\right]$

"Composite" Trapezoidal Rule   NOTE. For equally spaced nodes.

QUADRATIC INTERPOLATION

We are going to do the same process as we did for Linear Interpolation ... except now use $P_2(x)$.

- How many nodes do I need for $P_2(x)$? THREE!

use $\{a, c, b\}$ where $c = \frac{a+b}{2}$ so they are equally spaced.

1. Write down P_2

$$P_2(x) = \frac{(x-c)(x-b)}{(a-c)(a-b)}f(a) + \frac{(x-a)(x-b)}{(c-a)(c-b)}f(c) + \frac{(x-a)(x-c)}{(b-a)(b-c)}f(b)$$

$$= \frac{(x-c)(x-b)f(a)}{2h^2} + \frac{(x-a)(x-b)f(c)}{-h^2} + \frac{(x-a)(x-c)f(b)}{2h^2}$$

2. Plug in and Integrate

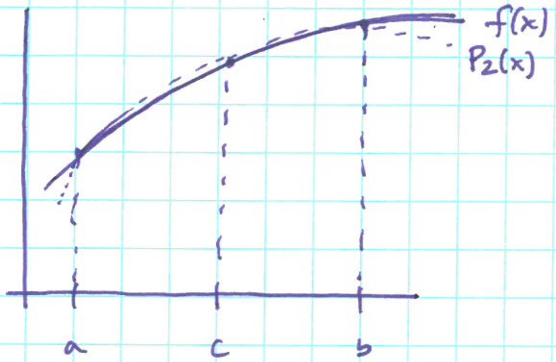
this can be messy!

$$\int_a^b P_2(x)\,dx = \frac{h}{3}\left[f(a) + 4f(c) + f(b)\right] \equiv S_2(f)$$

SIMPSONS RULE.

here we assumed equal spacing: $c - a = b - c = h$

and used $c = \frac{a+b}{h}$



We really just replaced the linear approximation with a quadratic one.

3. Generalize the rule to small subdivisions ... in otherwords consider piecewise quadratic interpolation.

in general $$\int_\alpha^\beta f(x)\,dx \approx \frac{\delta}{3}\left[f(\alpha) + 4f(\gamma) + f(\beta)\right]$$

where $\gamma = \frac{\alpha + \beta}{2}$

then if we divide up $[a, b]$; $h = \frac{b-a}{n}$ and

$x_j = a + jh \quad j = 0, 1, 2, \ldots$

$$I = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

and use simpsons on each sub-interval.

$$I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right] + \cdots$$

$$+ \frac{h}{3}\left[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

$$S_n(f) = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

"Composite" SIMPSONS RULE.