Numerical Analysis
Day 15.

Last Time: Approximate Integration.

1. By Hand – Use Taylor Polynomials to evaluate.
2. Computationally – Polynomial Interpolation to evaluate $\int_a^b f(x)\,dx$

LINEAR.

replace $f(x) \approx P_1(x) = \dfrac{(x-a)f(b) + (b-x)f(a)}{b-a}$ Linear interpolating polynomial.

DISCUSS

Integrate to get: $T_1 = \frac{1}{2}(b-a)[f(a)+f(b)]$

Trapezoidal Rule.

in general $\text{TRAP}(n) \equiv \int_a^{a+\Delta x} f(x)\,dx + \int_{a+\Delta x}^{a+2\Delta x} f(x)\,dx + \cdots \int_{b-\Delta x}^{b} f(x)\,dx$

Divide the region into n-sections.

SPECIAL CASE If we divide $[a,b]$ into equal pieces $h = \frac{b-a}{n}$

the we have points at $x_j = a + jh \quad j = 1,2,3,\ldots,n$

Then $\text{TRAP}(n) = T_n = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$

$$= \tfrac{1}{2}(x_1-x_0)[f(x_0)+f(x_1)] + \tfrac{1}{2}(x_2-x_1)[f(x_1)+f(x_2)] + \cdots + \tfrac{1}{2}(x_n-x_{n-1})[f(x_{n-1})+f(x_n)]$$

$$= \tfrac{1}{2}h\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Composite TRAPIZOIDAL RULE

From Code:

Ex $\int_0^1 e^{-x^2}\,dx$ when we decrease $h \to \frac{h}{2}$ h^2 convergence

Error goes from $E \to \frac{E}{4}$ Ratio of Errors.

QUADRATIC

Discuss

replace $$f(x) \approx P_2(x) = \frac{(x-c)(x-b)}{(a-c)(a-b)} f(a) + \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c) + \frac{(x-a)(x-c)}{(b-a)(b-c)} f(b)$$

Integrate to get: $$S_1 = \frac{h}{3}\left[f(a) + 4f(c) + f(b)\right]$$

where $c = \frac{a+b}{2}$ and $h = \frac{b-a}{2}$

In general $$SIMP(n) = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

where $x_j = a + jh \quad j = 1, 2, 3, \ldots n$

notice we need three points on each sub interval.

SPECIAL CASE $x_j = a + jh$ equal divisions.

Then $$SIMP(n) = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] + \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right] \cdots + \frac{h}{3}\left[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

From Code.

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) \cdots + 4f(x_{n-1}) + f(x_n)\right]$$

Ex $\int_0^1 e^{-x^2}\,dx$ when we decrease $h \to \frac{h}{2}$

Error goes from $E \to \frac{E}{16}$

ratio of errors.

h^4 convergence.

Today — a more formal discussion of error in TRAP(n) and SIMP(n).

Code trapezoidERR(n, 0, 1, 0.746824132812427, 0) notice order 2

Theorem Let $f(x)$ have two continuous derivatives on $[a,b]$ then

$$E_n^T(f) \equiv \int_a^b f(x)\,dx - T_n(f) = \frac{-h^2(b-a)}{12} f''(c_n)$$

for some $c_n \in [a,b]$.

The error in the trapezoid rule decreases roughly as h^2 ... second order convergence

Error is highly dependent on curvature!!

Let's prove this...

Consider the error over a single subdivision:

$$E = \int_\alpha^{\alpha+h} f(x)\,dx - \frac{h}{2}\left[f(\alpha) + f(\alpha+h)\right]$$

replaced $(b-a) = (\alpha+h-\alpha) = h$

Now expand $f(x)$ and $f(\alpha+h)$ in taylor polynomials about the point α

$$\underset{x \text{ near } \alpha}{f(x)} = f(\alpha) + (x-\alpha)f'(\alpha) + \frac{(x-\alpha)^2}{2} f''(c_n)$$

And when $x = \alpha + h$

$$f(\alpha+h) = f(\alpha) + hf'(\alpha) + \frac{h^2}{2} f''(c_n)$$

Expect order 2 convergence so stop after 2 terms, + error.

Now integrate $f(x)$

$$\int_\alpha^{\alpha+h} f(\alpha) + \cancel{h}^{(x-\alpha)} f'(\alpha) + \frac{\cancel{h^2}^{(x-\alpha)^2}}{2} f''(c_n)\,dx = xf(\alpha) + \tfrac{1}{2}(x-\alpha)^2 f'(\alpha) + \tfrac{1}{2}\tfrac{1}{3}(x-\alpha)^3 f''(c_n)\Big|_\alpha^{\alpha+h}$$

$$= (\alpha+h)f(\alpha) + \tfrac{1}{2}h^2 f'(\alpha) + \tfrac{1}{6}h^3 f''(c_n) - \alpha f(\alpha)$$

$$= hf(\alpha) + \tfrac{1}{2}h^2 f'(\alpha) + \tfrac{1}{6}h^3 f''(c_n)$$

Plugging this all back into E:

$$E = hf(\alpha) + \tfrac{1}{2}h^2 f'(\alpha) + \tfrac{1}{6}h^3 f''(c_n) - \frac{h}{2}\left[f(\alpha) + f(\alpha) + hf'(\alpha) + \frac{h^2}{2}f''(c_n)\right]$$

$$= \frac{h^3}{6}f''(c_n) - \frac{h^3}{4}f''(c_n) = -\frac{h^3}{12}f''(c_n)$$

Error on one division.

Over the whole domain...

$$E_n^T(f) = -\frac{h^3}{12}f''(c_1) + \cdots\cdots -\frac{h^3}{12}f''(c_n)$$

where c_n is some number in the j^{th} subinterval. $x_{j-1} \le c_j \le x_j$

$$= -\frac{h^3}{12}n\left[\underbrace{\frac{f''(c_1) + f''(c_2) + \cdots + f''(c_n)}{n}}_{\xi_n \text{ — average value}}\right]$$

by the intermediate value theorem

since $\displaystyle\min_{a \le x \le b} f''(x) \le \xi_n \le \max_{a \le x \le b} f''(x)$

and $f''(x)$ is continuous.

The average value is somewhere between the max and min.

There exists some number $\gamma_n \in [a,b]$ such that

$$f''(\gamma_n) = \xi_n$$

The function must go through the average value.

Also $hn = b - a$

Thus $$\boxed{E_n^T(f) = -\frac{h^2}{12}(b-a)f''(\gamma_n)}$$

How do we use this theorem?

Ex I want to approximate

$$I = \int_0^2 \frac{dx}{1+x^2} \quad \text{using } T_n(f).$$

what should I choose as n so that my error $E \le 5 \times 10^{-6}$?

Bound the error and solve for n.

$$E_n^T(f) = -\frac{h^2}{12}(b-a) f''(c_n)$$

$$f(x) = \frac{1}{1+x^2} \qquad f'(x) = \frac{-1}{(1+x^2)^2} \cdot 2x = \frac{-2x}{(1+x^2)^2}$$

$$f''(x) = \frac{-2(1+x^2)^2 + 2(1+x^2)^{\cancel{3}}\, 2x \cdot 2x}{(1+x^2)^4} = \frac{-2 + 8x^2(1+x^2)}{(1+x^2)^2}$$

$$= \cancel{\cdots}$$

$$\text{so} \quad |E_n^T| = \frac{h^2}{12}(b-a) f''(c_n) \le \frac{h^2}{12}(2)(2)$$

see this on the graph. desmos.com.

$\max\limits_{a \le x \le b} f''(c_n)$

$$\text{so} \quad |E_n^T| \le \frac{h^2}{3} \le 5 \times 10^{-6}$$

$$\text{or} \quad h \le .003873$$

$$\text{an} \quad n = \frac{2}{h} \ge 516.40 \qquad n = 517 \text{ would work.}$$

What was the hardest part of this calculation?
It is often difficult and messy to calculate $f''(x)$.
Then finding max $f''(x)$ is a pain...

We can discuss error in the Asymptotic sense... think about an approximate error.

Asymptotic Error

from our proof of the error formula.

Consider $E_n^T(f) = -\frac{h^3}{12}\left[f''(c_1) + f''(c_2) + \cdots + f''(c_n)\right]$

$$= -\frac{h^2}{12}\left[hf''(c_1) + hf''(c_2) + \cdots + hf''(c_n)\right]$$

in the limit as $h \to 0$ or as $n \to \infty$ this is a Riemann sum!

$$-\frac{h^2}{12} \lim_{h\to\infty} \sum_{i=1}^{n} h f''(c_i) = -\frac{h^2}{12}\int_a^b f''(x)\,dx$$

$$= -\frac{h^2}{12}\left[f'(b) - f'(a)\right]$$

so $E_n^T(f) \approx -\frac{h^2}{12}\left[f'(b) - f'(a)\right] \equiv \tilde{E}_n^T(f)$ asymptotic error estimate.

Often $f'(x)$ is easy to find ... This actually allows a correction to TRAP(n)!!

$$E_n^T(f) = I - T_n(f) \approx \tilde{E}_n^T(f)$$

so $I \approx T_n(f) + \tilde{E}_n^T(f)$

OR

$$I \approx T_n(f) - \frac{h^2}{12}\left[f'(b) - f'(a)\right]$$

The corrected trapezoid rule.

Ex $\int_0^1 e^{-x^2}\,dx$ the corrected trapezoid rule converges almost as well as SIMP(n)

using less points $\Rightarrow$ faster calculation.

code — add correction.

We can do a similar analysis for SIMPSONS RULE.

$$E_n^S(f) = \int_a^b f(x)\,dx - S_n(f) = \frac{-h^4(b-a)}{180} f^{(4)}(c_n)$$

assuming $f(x)$ has four continuous derivatives.

$a \le c_n \le b$.

Use this the same way
Bound error solve for h/or n

Asymptotically $\tilde{E}_n^S(f) \equiv \frac{-h^4}{180}\left[f'''(b) - f'''(a)\right]$

we can do a corrected simpsons rule — but calculating $f'''(x)$ is a real pain... often not worth it.

$$I \approx S_n(f) - \frac{h^4}{180}\left[f'''(b) - f'''(a)\right]$$

<u>Ex</u> $\int_0^1 e^{-x^2}\,dx$ this does improve SIMP but not as much as TRAP.

why? Smaller Correction $\frac{h^4}{180}$

$$f'''(x) = -4e^{-x^2}(2x^3 - 3x)$$

$n=4$ then $h = 1/4$

code
add correction.

and $\tilde{E}_n^S(f) = \frac{-(1/4)^4}{180} \cdot (-4e^{-1}(-1))$

$\sim -.000032$