Day 17
Numerical Analysis.

Interesting Ideas and Effects of Error.

DIFFERENTIATION.

last time: $D_h^2 f(x_1) = \dfrac{f(x_2) - 2f(x_1) + f(x_0)}{h^2} \approx f''(x_1)$

we discussed the error in the <u>approximation</u>
but what about error in the <u>data</u> or <u>computer</u> rounding?

Introduce the idea of data errors

$$\epsilon_i = \underbrace{f(x_i)}_{\text{the real } f(x) \text{ value}} - \underbrace{\hat{f}_i}_{\text{the data used.}} \qquad i = 0, 1, 2$$

so computationally we get

$$\hat{D}_h^2 f(x_1) = \frac{\hat{f}_2 - 2\hat{f}_1 + \hat{f}_2}{h^2}$$

how much did we lose by using $\hat{f}$?

well $\hat{f}_i = f(x_i) - \epsilon_i$ just plug this in!

The method Gives....

$$\hat{D}_h^2 f(x_1) = \frac{f(x_2) - \epsilon_2 - 2f(x_1) + 2\epsilon_1 + f(x_2) - \epsilon_0}{h^2}$$

So the error is...

$$\text{Error} = f''(x_1) - \hat{D}_h^2 f(x_1) = f''(x_1) - \frac{f(x_2) - 2f(x_1) + f(x_2)}{h^2} + \frac{\epsilon_2 - 2\epsilon_1 + \epsilon_0}{h^2}$$

$$\approx \underbrace{-\frac{h^2}{12} f^{(4)}(x_1)}_{\text{we proved this last time w/ Taylor Polynomials}} + \underbrace{\frac{\epsilon_2 - 2\epsilon_1 + \epsilon_0}{h^2}}_{\text{error propigation from data.}}$$

The errors ϵ_2, ϵ_2, and ϵ_0 are random on some interval $[-\delta, \delta]$.

- Experimental δ is the bound on experiment/measurement error
- Computational δ is the bound on rounding and chopping error.

So $\left| f''(x_1) - \hat{D}_h^2 f(x_1) \right| \leq \frac{h^2}{12} \left| f^{(4)}(x_1) \right| + \frac{4\delta}{h^2} \longrightarrow$ Eg. $\epsilon_2 = \delta$, $\epsilon_1 = -\delta$, $\epsilon_2 = \delta$

we took the worst case for δ's in each ϵ_i.

- So as $h \to 0$ what happens?

At some point decreasing h will increase the error ... this depends on how small δ is.

Q's on numerical Differentiation?

INTEGRATION

Recall: Asymptotic Error for TRAP and SIMP...

$$E_n^T(f) \approx \tilde{E}_n^T(f) \equiv -\frac{h^2}{12}\left[f'(b) - f'(a)\right]$$

$$E_n^S(f) \approx \tilde{E}_n^S(f) \equiv -\frac{h^4}{180}\left[f'''(b) - f'''(a)\right]$$

provided the derivatives exist!

Use this to come up with an improved method...

We could write either formula in the form:

$$\tilde{E}_n(f) = \frac{C}{n^p}$$

C = some constant
n = # of points
p = exponent.

For TRAP

$$C = -\frac{(b-a)^2}{12}\left[f'(b) - f'(a)\right] \quad \leftarrow h = \frac{(b-a)}{n}$$

and $p = 2$ ✓

so $n = \frac{(b-a)}{h}$

For many integration formulas we can write the asymptotic error in this form.

LETS USE THIS TO FIND POSSIBLY BETTER APPROXIMATIONS...

$$I - I_n \approx \frac{C}{n^p} \rightarrow \text{approximate/asymptotic error}$$

I: the real value of integral

I_n: computed value using one of our integration rules.

We could also write $I - I_{2n} \approx \dfrac{C}{2^p n^p}$

I_{2n}: double the # of points.

Putting these together (compare $I - I_n$ and $I - I_{2n}$) (factor of 2^p)

$$I - I_n \approx 2^p\left[I - I_{2n}\right]$$

check... $2^p\left[\dfrac{C}{2^p n^p}\right] = \dfrac{C}{n^p}$ ✓

Solve for I ... to get an approximation for my integral...

$$I(1-2^P) \approx I_n - 2^P I_{2n}$$

$$I \approx \frac{I_n - 2^P I_{2n}}{(1-2^P)} = \frac{2^P I_{2n} - I_n}{2^P - 1} = I_{2n} + \frac{I_{2n} - I_n}{2^P - 1}$$

write as iteration or improvement

Richardson's extrapolation

$$\boxed{I \approx I_{2n} + \frac{I_{2n} - I_n}{2^P - 1}}$$

so with the TRAPEZOID RULE ... you calculate

T_{2n} and T_n then use

$$I \approx T_{2n} + \frac{T_{2n} - T_n}{3} \qquad \text{because } P = 2$$

What would the formula be for simpsons? $P = 4$

$$I \approx S_{2n} + \frac{S_{2n} - S_n}{15}$$

This can also be used to estimate error...

$$I - T_{2n} \approx \frac{T_{2n} - T_n}{3}$$

The right term approximates the error in T_{2n}.

While loop when calculating the integral... reduce $n \to 2n$ until error is reached.

<u>AITKEN EXTRAPOLATION</u>

again start with $I - I_n \approx \frac{C}{n^P}$

This time compute I_n, I_{2n} and I_{4n} ... mwahhaha Richardson ... I'm adding an extra term!

$$I - I_n \approx \frac{C}{n^P} \qquad I - I_{2n} \approx \frac{C}{2^P n^P}$$

$$I - I_{4n} \approx \frac{C}{4^P n^P}$$

Consider the ratio of these errors...

$$\frac{I - I_n}{I - I_{2n}} \approx 2^p \quad \text{and} \quad \frac{I - I_{2n}}{I - I_{4n}} \approx 2^p$$

so $$\frac{I - I_n}{I - I_{2n}} \approx \frac{I - I_{2n}}{I - I_{4n}}$$ — criss cross applesauce!

and $$(I - I_n)(I - I_{4n}) \approx (I - I_{2n})^2$$

NOW SOLVE FOR I ...

$$\cancel{I^2} - (I_n + I_{4n})I + I_n I_{4n} \approx \cancel{I^2} - 2I_{2n}I + I_{2n}^2$$

$$I\left[I_n + I_{4n} - 2I_{2n}\right] \approx I_n I_{4n} - I_{2n}^2$$ flipped signs...

so $$I \approx \frac{I_n I_{4n} - I_{2n}^2}{I_n + I_{4n} - 2I_{2n}}$$ → add and subtract I_{4n} then simplify

we rewrite this as

Aitkin Extrapolation → $$\boxed{I \approx I_{4n} - \frac{(I_{4n} - I_{2n})^2}{(I_{4n} - I_{2n}) - (I_{2n} - I_n)}}$$

we rewrite this because in the other form we could get loss of significance errors. usually better to write things in an "update" form.

Often this will significantly improve convergence BUT you need to first calculate I_n, I_{2n}, I_{4n}.

NOTES ON PERIODIC FUNCTIONS.

If $f(x+\tau) = f(x) \quad -\infty < x < \infty$
for some smallest number τ
Then we call $f(x)$ periodic w/ period τ

Derivatives of periodic functions are also periodic

$$f^{(m)}(x+\tau) = f^{(m)}(x)$$

we run into periodic functions all the time in nature!

If we are trying to calculate

$$I(f) = \int_a^b f(x)\,dx \quad \text{for } f(x) \text{ periodic}$$

then we can run ~~into~~ into some special cases...

IF $(b-a)$ is an integer multiple of the period then consider what happens to the asymptotic error in the TRAP rule:

$$\tilde{E}_n^T(f) \equiv -\frac{h^2}{12}\left[f'(b) - f'(a)\right] = 0$$

because $f'(b) = f'(a)$

This means that the TRAP will have improved convergence ... the method will perform better!

We find that if $f(x)$ is periodic with $b-a$ an integer multiple of the period τ then $I - T_n$ decreases RAPIDLY ...

The trapezoid rule is optimal in these cases!