Numerical Analysis.

Today we start DIFFERENTIAL EQNS.

Goal: Given $\frac{dy}{dx} = y' = f(x,y)$
with some initial condition
$y(x_0) = y_0$

find the soln $y(x)$ on some interval
$x_0 \le x \le b$, assuming $y(x)$ exists
and is unique.

BACKGROUND.

ORDER $\rightarrow$ the highest derivative

$y' = f(x,y)$ is first order

$y'' + 2y' = 0$ is <u>second</u> order

- We focus on FIRST ORDER because all higher order ODE's can be expressed as an equivalent system of first order eqns.

ODE vs PDE
$\downarrow$ ordinary, $\downarrow$ partial
ODE $\rightarrow$ one independent var
PDE $\rightarrow$ more than one.

LINEAR vs NONLINEAR — if the dependent variable and its derivatives <u>do not</u> appear as a function then the eqn is linear.

$y' = 3y - \sin(x)$ vs. $y' = 3x - \sin(y)$.

nonlinear equns have more issues with existence and uniqueness.

First some simple examples and solution methods.

Ex The most simple case:

$$\left.\begin{aligned} y' &= f(x) \\ y(x_0) &= y_0 \end{aligned}\right\rangle$$ integrate for the soln

$$y(x) = \int f(x)\,dx + C$$

SOLVE $\left.\begin{aligned} y' &= \sin(x) \\ y(0) &= 3 \end{aligned}\right\rangle$ $y(x) = -\cos(x) + C$ general soln

$$y(0) = 3 = -\cos(0) + C$$
$$C = 4$$

particular soln:
$$y(x) = -\cos(x) + 4.$$

TO SHOW SOMETHING IS A SOLN ... PLUG IN !!

in otherwords just take the derivatives!

Ex The linear case

$$\left.\begin{aligned} y' &= a(x)y + b(x) \\ y(x_0) &= y_0 \end{aligned}\right\}$$ Method of ~~undetermined coef~~ integrating factor will work.

SOLVE $y' = \lambda y + b(x)$ $\quad \lambda = \text{constant}$
$y(x_0) = y_0$

define the integrating factor $p(x) = e^{-\lambda x}$

this allows us to rewrite the eqn

$$\frac{d}{dx}\left[e^{-\lambda x} y\right] = e^{-\lambda x} b(x)$$ show that this works.

then we can integrate and solve for y...

$$y = ce^{\lambda x} + \int_{x_0}^{x} e^{-\lambda(t-x)} b(t)\,dt.$$

$$c = e^{-\lambda x_0} y(x_0)$$

Ex The Seperable Case (NONLINEAR)

$$y' = g(y) \cdot f(x)$$

$$\frac{dy}{dx} = g(y) \cdot f(x) \rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$$

again we integrate and solve for $y(x)$.

PROBLEM... all of these methods rely on integration. Integrals can be hard or impossible and often we can't solve for a closed form.

NUMERICAL METHODS ... allow us to solve these ODEs with out integration.

Before considering the wide range of Numerical Methods we need to discuss SOLVABILITY and STABILITY.

Solvability Theory — Does a soln exist and if it does exist is the soln unique.

Consider a function $y(x)$ that satisfies

$$y' = f(x,y) \quad \text{for} \quad x \geq x_0 \quad y(x_0) = y_0$$

assume there is some open set D that is a ~~the~~ subset of the xy-plane and contains (x_0, y_0) for which:

1. If two points (x,y) and (x,z) are contained in D then the line segment joining them is contained in D.
2. $f(x,y)$ is continuous for all points (x,y) in D
3. $f_y(x,y)$ is continuous for all points (x,y) in D

THEN there is an interval $[c,d]$ containing x_0 and a unique function $y(x)$ defined on $[c,d]$ which satisfies the differential eqn.

Ex $y' = xy^2$, $y(0) = 1$ here $f(x,y) = xy^2$, $f_y(x,y) = 2xy$

These are continuous on in a neighborhood near x_0.
So we can say a soln exists on some interval $[c,d]$.

Ex $y' = \frac{x}{y}$, $y(1) = 0$ $f(x,y) = \frac{x}{y}$, $f_y(x,y) = -\frac{x}{y^2}$

Near $(1,0)$ these are BOTH discontinuous
no soln exists, near $(1,0)$.

Ex $y' = \lambda y + b(x)$, $\lambda =$ constant, $y(0) = 1$ $f(x,y) = \lambda y + b(x)$, $f_y(x,y) = \lambda$

These are continuous in a neighborhood near x_0. Soln exists on some interval $[c,d]$.

How can we better identify the interval $[c,d]$.

<u>THE LIPSCHITZ CONDITION</u> – if the f_y condition is satisfied then
There is a non-negative constant K for which

$$|f(x,y) - f(x,z)| \le K|y-z|$$

for all points (x,y) and (x,z) in the region D.

OR in practice we test

$$K = \max_{(x,y)\in D} \left| \frac{\partial f(x,y)}{\partial y} \right|$$

if K is finite then the soln exists and is unique.

We actually often say

$$K = \max_{\substack{-\infty < y < \infty \\ x_0 \le x \le b}} \left| \frac{\partial f(x,y)}{\partial y} \right| < \infty$$

Because we are solving on some finite region.

How does this help us ...

Consider our examples.

Ex $y' = xy^2$, $y(0) = 1$ $\quad K = \max\limits_{\substack{-\infty < y < \infty \\ x_0 \le x \le b}} |2xy|$

this is not less than ∞ because $y \to \pm\infty$. Near $(0,1)$ we are okay... $f_y(0,1) = 0$ there will be limitations.

The soln to this ODE is...

$$y(x) = \frac{1}{1 - \frac{1}{2}x^2}$$

we see that this soln is discontinuous outside $-\sqrt{2} < x < \sqrt{2}$.

Ex $y' = \lambda y + b(x)$

$$K = \max_{\substack{-\infty < y < \infty \\ x_0 \le x \le b}} |\lambda|$$

this is bounded for all ranges! So the soln exists, is unique, no discontinuities ... on $-\infty < x < \infty$.

Once we have Existence and Uniqueness ... we now worry about Stability.

IF we make a small change in the data how much does that change our final soln.

- If a small change in the data causes a LARGE change in the final soln the Our ODE is

UNSTABLE or ILLCONDITIONED.

otherwise it is STABLE or WELL-CONDITIONED.

When using a numerical method ... this is important to test!

<u>Ex</u> Consider $y' = 100y - 101e^{-x}$
$y(0) = 1$

The soln is $y(x) = e^{-x}$ check this by plugging in!

- we would solve using an integrating factor.

If we Perturb the Initial Condition

$$y(0) = 1 + \epsilon$$

the soln changes to $y_\epsilon = e^{-x} + \epsilon e^{100x}$

This Problem is UNSTABLE ... we have to be very careful in our initial condition to get a correct soln.

There are ways of testing this before programming ... but you can also carry out a stability test.