Numerical Analysis.

Numerical Methods For Ode's.

Goal: Solve the ode $y' = f(x,y)$ $x_0 \le x \le b$
$y(x_0) = y_0$

on a set of node points: $x_0, x_1, x_2 \dots x_n \le b$
often we will use equally spaced nodes:

$$x_i = x_0 + ih \qquad i = 0, 1, \dots, N$$

So our solution is a discret set of values

$$y_h(x_i) \equiv y_i \approx Y(x_i)$$

we will use capital Y to denote the true soln.

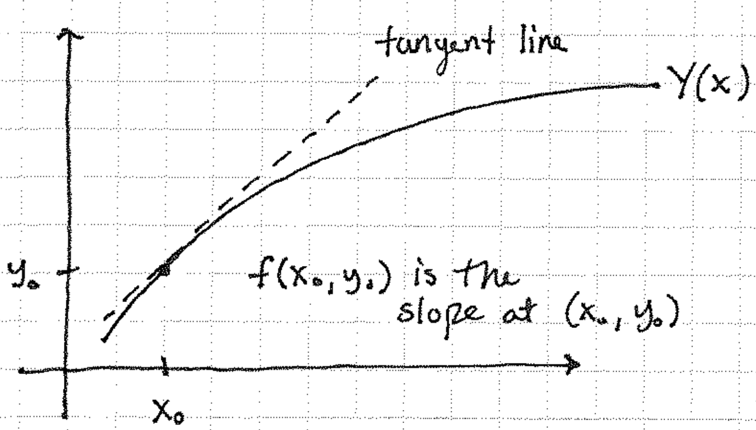
Most simple and straight forward method...

EULERS METHOD:

$$y_{n+1} = y_n + h f(x_n, y_n) \qquad n = 0, 1, 2, \dots N-1$$

we can derive this from multiple perspectives.

Geometrically



Linear approximation.

Use the tangent line to approximate $Y(x)$ a small, h away.

$y_1 = y_0 + h f(x_0, y_0)$
this steps us forward
this is Eulers Method.

How could we improve this?

<u>Taylor Series</u> approximate $y(x_0+h)$ using the Taylor Polynomial

$$y(x_0+h) \approx y(x_0) + h y'(x_0) + \frac{h^2}{2!} y''(x_0) + \cdots + \frac{h^p}{p!} y^p(x_0)$$

for eulers method $p=1$

$$Y(x_0+h) \approx y_1 = y(x_0+h) \approx y(x_0) + h y'(x_0) = y(x_0) + h f(x_0, y_0).$$

This is Eulers Method

From this we can also talk about error...

TAYLOR REMAINDER

$\frac{h^2}{2} Y(\xi) \longrightarrow$ $\rightarrow$ where $x_0 \le \xi_0 \le x_1$

We will use this later when we discuss general and asymptotic error.

this is the Taylor Remainder when $p=1$

<u>Numerical Differentiation</u>

By The ODE...

$$y'(x_n) \approx \frac{y(x_n+h) - y(x_n)}{h} = f(x_n, y_n)$$

solve for $y(x_n+h) = y(x_{n+1})$

this leads to eulers method.

Be very careful when using numerical derivatives and inventing new methods. Many formulas will lead to poor soln methods, always check error and convergence.

How do we use Eulers Method in practice?

<u>Ex</u> Solve $y' = \frac{y + x^2 - 2}{x+1}$ $\quad y(0) = 1$

#1 Check existence and uniqueness

$$f(x,y) = \frac{y + x^2 - 2}{x+1} \qquad f_y(x,y) = \frac{1}{x+1}$$

$$K = \max_{\substack{-\infty < y < \infty \\ x_0 \le x \le b}} \left| \frac{1}{x+1} \right| < \infty$$

because $x_0 = 0$ no problems for the given initial condition.

EXAMPLE. Solve

$$Y'(x) = \frac{Y(x) + x^2 - 2}{x + 1}, \qquad Y(0) = 1$$

The solution is

$$Y(x) = x^2 + 2x + 2 - 2\,(x+1)\log\,(x+1)$$

We give selected results for three values of h. Note the behaviour of the error as h is halved.

h	x	$y_h(x)$	*Error*	*Relative Error*
0.2	1.0	2.1592	$6.82E-2$	0.0306
	3.0	5.4332	$4.76E-1$	0.0805
	5.0	14.406	1.09	0.0703
0.1	1.0	2.1912	$3.63E-2$	0.0163
	3.0	5.6636	$2.46E-1$	0.0416
	5.0	14.939	$5.60E-1$	0.0361
0.05	1.0	2.2087	$1.87E-2$	0.00840
	3.0	5.7845	$1.25E-1$	0.0212
	5.0	15.214	$2.84E-1$	0.0183

#2. Develop Eulers Method

$$y_{n+1} = y_n + h\left[\frac{y_n + x_n^2 - 2}{x_n + 1}\right]$$

#3 Iterate using the method... grid or node points:

$x_n = [0, h, 2h, \ldots, (n-1)h, b]$

you specify $h = \frac{b-a}{n}$ and b.

$y_0 = 1$

$y_1 = 1 + h\left[\frac{1 + 0^2 - 2}{0+1}\right] = 1 - h$ because $y_0 = 1$ $x_0 = 0$

$y_2 = (1-h) + h\left[\frac{(1-h) + h^2 - 2}{h+1}\right]$ because $y_1 = 1-h$ $x_1 = h$

$\vdots$

#4 Plot the final soln.

plot (x_n, y_n) $\quad y_n = [y_0, y_1, y_2 \ldots y_n]$

Show .pdf slide... What happens to the Error?

1. as h get smaller the error is smaller
2. as we march out farther the error can get bigger.

WRITE PSUEDO CODE FOR EULERS METHOD

NOTE ... what do you need to specify before you start iterating?
I often have a separate function for $f(x,y)$ so I don't actually re-do #2 every time making my code more general.

ERROR ANALYSIS.

$$\underbrace{Y(x_{n+1})}_{\text{use taylor series}} - \underbrace{y_{n+1}}_{\text{use eulers method}} = \left[Y(x_n) + hY'(x_n) + \frac{h^2}{2}Y''(\xi_n)\right] - \left[y_n + hf(x_n, y_n)\right]$$

$$= \left[Y(x_n) + h\,f(x_n, Y(x_n)) + \frac{h^2}{2}Y''(\xi_n)\right] - \left[y_n + hf(x_n, y_n)\right]$$

$$= (Y(x_n) - y_n) + h\left[f(x_n, Y(x_n)) - f(x_n, y_n)\right] + \frac{h^2}{2} Y''(\xi_n)$$

From the mean value thm

$$f(x_n, Y(x_n)) - f(x_n, y_n) = \frac{\partial f(x_n, \zeta_n)}{\partial y}\left[Y(x_n) - y_n\right]$$

looks dangerously close to the LIPSCHITZ CONDITION

for some ζ_n between $Y(x_n)$ and y_n

call $e_h(x) \equiv Y(x) - y_h(x)$ the error at x. for approx with h step size.

$$e_h(x_{n+1}) = \left[1 + h\frac{\partial f(x_n, \zeta_n)}{\partial y}\right] e_h(x_n) + \frac{h^2}{2} Y''(\xi_n)$$

for now on assume $K \equiv \max_{\substack{-\infty < y < \infty \\ x_0 \le x \le b}} \left|\frac{\partial f(x,y)}{\partial y}\right| < \infty$

hmm familiar !!
we need this or error $\to \infty$
no soln.

<u>CONSIDER EQNS FOR WHICH</u>

$$\frac{\partial f(x,y)}{\partial y} \le 0 \qquad x_0 \le x \le b \quad -\infty < y < \infty$$

WELL-CONDITIONED.

Then $\quad -1 \le 1 + h\frac{\partial f(x,y)}{\partial y} \le 1 \quad$ as long as h is small.

$$h < \frac{1}{K}$$

this is because we want $\left[1 + h\frac{\partial f(x_n, \xi_n)}{\partial y}\right]$ to decrease $e_n(x_n)$!! or at least not magnify it !

is sufficiently small. Why? $\frac{1}{K}\frac{\partial f(x,y)}{\partial y}$ is negative and may < 1.

$$|e_h(x_{n+1})| \le |e_n(x_n)| + \frac{h^2}{2}\underbrace{\max_{x_0 \le t \le b}|Y''(t)|}_{\|Y''\|_\infty}$$

to BOUND THE ERROR.

or

$$|e_h(x_n)| \le \frac{h}{2}(x_n - x_0)\|Y''\|_\infty$$

$$|Y(x_n) - y_n| \le c(x_n)\,h$$

for some number $c(x_n)$ that depends on $\frac{(x_n - x_0)}{2}\|Y''\|_\infty$

If we know $Y''(x)$ we can estimate error and choose an exact h that would keep the error sufficiently small...

Problem... we need $Y(x)$ the true soln to do this!

ASYMPTOTIC ERROR.

"ORDER h^2" terms bounded by a constant times h^2

$$Y(x_n) - y_h(x_n) = D(x_n)h + O(h^2)$$

I state but do not prove this

IDEA...

if $|Y(x_n) - y_n| \le c(x_n)h$ then there must exist a function $D(x_n)$ so that this is exact. $D(x_n)$ depends on Y and its derivatives.

asymptotically speaking... for small h...

$$Y(x_n) - y_h(x_n) \approx D(x_n)h$$

in practice we cannot know $D(x_n)$

BUT THIS IDEA LEADS TO RICHARDSON EXTRAPOLATION.

if we are solving $y' = f(x,y)$ $x_0 \le x \le b$ $-\infty < y < \infty$ with eulers method then asymptotically

TWO DIFFERENT EULER APPROX

$$Y(x) - y_h(x) \approx D(x)h$$
$$Y(x) - y_{2h}(x) \approx D(x)2h$$

cut h in half.

mult 1^{st} eqn by 2 and subtract

$$Y(x) - 2y_h(x) + y_{2h}(x) \approx 0$$

solve for $Y(x) \approx 2y_h(x) - y_{2h}(x)$ Richardson's Extrapolation.

In Practice [illegible] at each point.

USE THIS TO ESTIMATE ERROR...

$$Y(x) - y_h(x) \approx y_h(x) - y_{2h}(x)$$

Ex. $h = .1$ $2h = .2$

Calculate euler for both

tells error for $h = .1$