Numerical Analysis - 26
Review of Linear Systems.

Today - Linear Systems Review.

1. How do we represent a linear system of equations as a matrix.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

↑ ↑ ↑
line up like variables using zeros if needed. $\Rightarrow$ read off the A matrix.

General Form: $Ax = b$

WORKSHEET.

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2. Classify Linear Systems ... in otherwords how complicated are they to solve?

<u>SIZE</u> small systems $\rightarrow$ solve by hand, row reduction, inverse by hand.
moderate systems $\rightarrow$ matrix inverse on computer:

$$x = A^{-1}b.$$

large systems $\rightarrow$ even using a computer matrix inverse takes a long time. Need more advanced methods / tricks.

<u>SPARSE vs DENSE</u> a sparse matrix contains lots of zeros. For example a large tridiagonal matrix contains mostly zeros - we often do not store the zeros.
a dense matrix is more complicated has few zeros and we must store the whole thing.

DIRRECT SOLUTION METHODS. - methods with a finite # of steps resulting in an exact solution.
Gaussian Elimination

ITERATION METHODS - iterate to an approximate solution, most common with large sparse systems.
PDE's is an area of active research!!

Homogeneous Linear Systems $Ax = 0$
when b contains only zeros.

Theorem - The following are equivalent statements.

1. For each b, there is exactly one soln $\vec{x}$
2. For each b a solution exists
3. THE associated homogeneous system $A\vec{x} = 0$ has only the zero solution $\vec{x} = \emptyset$
4. $\text{Det}(A) \neq 0$
5. The inverse, A^{-1}, exists.

These are all ways to show the the solution or inverse exist.
If one is true they are all true.

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3. FreeMat Code: and Important Matrices.

We can use freemat to set up very complicated matrices... we have done some of this.

The Zero Matrix $A + \emptyset = A$ or $A - A = \emptyset$

$$\emptyset = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ contains only zeros

The Identity Matrix $AI = A$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ones along the diagonal.

Matrix Inverse: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \dfrac{1}{\det(A)}\begin{bmatrix} d & -c \\ -c & a \end{bmatrix}$

For the 2×2 it is simple...

$B = \begin{bmatrix} 3 & -2 \\ 7 & 1 \end{bmatrix} \qquad B^{-1} = \dfrac{1}{3+14}\begin{bmatrix} 1 & 2 \\ -7 & 3 \end{bmatrix} = \dfrac{1}{17}\begin{bmatrix} 1 & 2 \\ -7 & 3 \end{bmatrix}$

For larger systems it is more complicated.

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

the Adjugate = the transpose of the cofactor matrix.

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

First the determinate: $\det(A) = (1)\det\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} + -(2)\det\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + (3)\det\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$

$$(\text{cofactor})^T = \begin{bmatrix} +\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} & -\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} & +\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

This is computationaly intense ... even for the computer if A is very large!

LOTS OF TRICKS

BASIC OPERATIONS WORKSHEET ... do by hand use FreeMat to check.

ELEMENTARY ROW OPERATIONS

As preparation for the discussion of Gaussian Elimination in Section 6.3, we introduce three elementary row operations on general rectangular matrices. They are:

i) *Interchange of two rows.*

ii) *Multiplication of a row by a nonzero scalar.*

iii) *Addition of a nonzero multiple of one row to another row.*

Consider the rectangular matrix

$$A = \begin{bmatrix} 3 & 3 & 3 & 1 \\ 2 & 2 & 3 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

We add row 2 times (-1) to row 1, and then add row 3 times (-1) to row 2 to obtain the matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Add row 2 times (-1) to row 1, and to row 3 as well:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

Add row 1 times (-2) to row 3:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Interchange row 1 and row 2:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Finally, we multiply row 3 by $1/3$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$$

This is obtained from A using elementary row operations. A reverse sequence of operations of the same type converts this result back to A.

MATRICES in MATLAB

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In *MATLAB*, A can be created as follows.

$$\begin{aligned}
&A = [1 \quad 2 \quad 3; \quad 2 \quad 2 \quad 3; \quad 3 \quad 3 \quad 3]; \\
&A = [1,\ 2,\ 3;\ 2,\ 2,\ 3;\ 3,\ 3,\ 3]; \\
&A = [1 \quad 2 \quad 3 \\
&\qquad\ \ 2 \quad 2 \quad 3 \\
&\qquad\ \ 3 \quad 3 \quad 3];
\end{aligned}$$

Commas can be used to replace the spaces. The vector b can be created by

$$b = ones(3, 1);$$

Consider setting up the matrices for the system $Ax = b$ with

$$A_{i,j} = \max\{i, j\}, \qquad b_i = 1, \qquad 1 \le i, j \le n$$

One way to set up the matrix A is as follows:

$$\begin{array}{ll} A = zeros(n, n); & \\ \textbf{for} & i = 1 : n \\ & A(i, 1 : i) = i; \\ & A(i, i + 1 : n) = i + 1 : n; \\ \textbf{end} & \end{array}$$

and set up the vector b by

$$b = ones(n, 1);$$

PARTITIONED MATRICES IN MATLAB

In MATLAB, matrices can be constructed using smaller matrices. For example, let

$A = [1, \quad 2; \quad 3, \quad 4]; \quad x = [5, \quad 6]; \quad y = [7, \quad 8]';$

Then

$$B = [A, \quad y; \quad x, \quad 9];$$

forms the matrix

$$B = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 4 & 8 \\ 5 & 6 & 9 \end{bmatrix}$$

Basic Matrix Operations — in-class worksheet

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -2 \\ 7 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A is a 2×2 matrix, with entries:

$$a_{11} = 2 \qquad a_{12} = 1$$
$$a_{21} = -1 \qquad a_{22} = 4$$

Compute:

$$A + B = \begin{bmatrix} 5 & -1 \\ 6 & 5 \end{bmatrix} = B + A, \qquad 3A = \begin{bmatrix} 6 & 3 \\ -3 & 12 \end{bmatrix}$$

The column vectors of A are:

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

The transpose of A is

$$A^T = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

The products AB and BA are not equal:

$$AB = \begin{bmatrix} 13 \\ 6 \end{bmatrix}, \qquad BA = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

A times a vector gives a linear combination of the columns of A:

$$A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ -c_1 + 4c_2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Multiplying by the identity leaves a matrix unchanged:

$$AI = \begin{bmatrix} \quad \end{bmatrix} = A, \qquad IA = \begin{bmatrix} \quad \end{bmatrix} = A$$

The inverse of A is

$$A^{-1} = \begin{bmatrix} \quad \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

Check:

$$A^{-1}A = \begin{bmatrix} \quad \end{bmatrix} = I, \qquad AA^{-1} = \begin{bmatrix} \quad \end{bmatrix} = I$$

To compute the determinant of the 3×3 matrix

$$C = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 1 & 4 \\ -2 & 5 & 1 \end{bmatrix}$$

we first find the three submatrices

$$C_{11} = \begin{bmatrix} 1 & 4 \\ 5 & 1 \end{bmatrix}, \qquad C_{12} = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}, \qquad C_{13} = \begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix},$$

and then use the cofactor expansion across the first row:

$$\det(C) = 2\det(C_{11}) - 4\det(C_{12}) + (-1)\det(C_{13}) =$$

$$= 2(-19) - 4(11) + (-1)(17) =$$