Numerics Day 2.

How did programming go? Homework?

Added a portion to HW1b.

Last time we showed how to derive

$$P_n(x) = \sum_{j=1}^{n} \frac{(x-a)^j}{j!} f^{(j)}(a)$$

Taylor Polynomials.

To expand a function (or approximate a function) near $x=a$ we

1. Use the formula to get $P_n(x)$ near a where we choose n to be large enough for a "good" approximation
2. Then to use this we say $f(b) \sim P_n(b)$ as long a b is near a.

How "good" is good enough?

$$\text{ERROR} = \left| \begin{matrix}\text{true} \\ \text{exact} \\ \text{value}\end{matrix} - \begin{matrix}\text{approximate} \\ \text{value}\end{matrix} \right|$$

last time we said $e^{.01} \sim 1.01005$

What is the ERROR?

true exact value = ??? um we don't really know!

How do we deal w/ error when we don't know the true value?

Often we find a way to BOUND the error... we can say about how bad the error is.

In good cases we can find an exact expression for the error, but it contains an unknown.

TAYLORS REMAINDER...

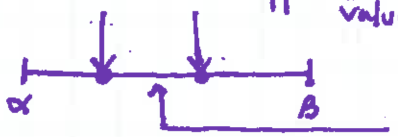
Assume $f(x)$ has $n+1$ continuous derivatives on the interval $\alpha < x < \beta$ and that the point a belongs to that interval. For the Taylor polynomial $P_n(x)$ let $R_n(x) \equiv f(x) - P_n(x)$ denote the remainder error in approximating $f(x)$ by $P_n(x)$. Then

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c_x)$$

for c_x an unknown point between a and x.

point where we expanded a

x point where we need an approximate value.



c_x is somewhere in here. This comes from the mean value theorem: Appendix A.

we can take all the derivatives in this range and construct $P_n(x)$ near $x=a$

Let $f(x)$ be continuous for $a \le x \le b$ and differentiable for $a < x < b$. Then there is at least one point c in the range $a < x < b$ for which

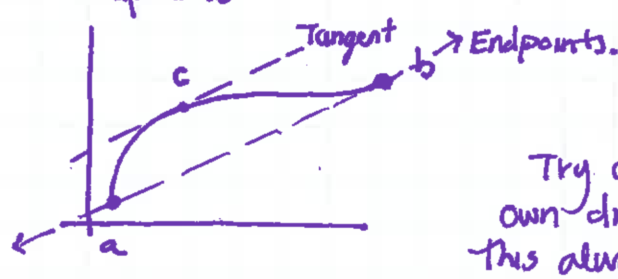
$$f(b) - f(a) = f'(c)(b-a)$$

We will use MVT a lot this semester.

$$\frac{f(b)-f(a)}{(b-a)} = f'(c)$$

slope of line connecting the endpoints

slope of tangent line at $x=c$



Try on your own drawing this always works :)

How do we use this?

Ex $f(x) = e^x$ near $x = 0$

$$e^x \sim 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}$$

Now use this to approximate $f(1) = e^1 \sim 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{x^n}{n!}$

How many terms do we need? Consider the remainder

$$e^1 - P_n(1) = R_n(1)$$

$$R_n(x) = \frac{(x-0)^{n+1}}{(n+1)!} e^{c_x}$$ where c_x is between x and $a=0$

$$R_n(1) = \frac{e^{c_x}}{(n+1)!}$$ where $0 < c_x < 1$.

Now we can investigate the error... what if we kept $n=3$ terms?

Q1 I have $n=3$ terms what is the error.

$$R_3(1) = \frac{e^{c_x}}{4!} = \frac{e^{c_x}}{24} \leq \frac{e^1}{24} < \frac{3}{24} = 0.125$$

BOUND The error

This says → with $n=3$ terms the error in the taylor polynomial approximation is less than 0.125.

Q2 I want my error to be less than — how many terms do I need.

What if I need my error to be less than 10^{-9}

$$\frac{e^{c_x}}{(n+1)!} \leq \frac{e^1}{(n+1)!} < \frac{3}{(n+1)!}$$

here I am chosing the largest value for my remainder with c_x between a and the point where I am approximating.

WORST CASE SCENARIO!

<u>So Let's Solve</u>

$$\frac{3}{(n+1)!} \leq 10^{-9}$$

Solve or guess and check!

$n = 10$: $3/11! \cong 7.5 \times 10^{-8}$

$n = 11$: $3/12! \cong 6.2 \times 10^{-9}$ $\quad$ $n=12$: $3/13! \cong 4.82 \times 10^{-10}$

Example. $f(x) = \cos(x^2)$

expand in Taylor Polynomial about $x=0$ and keep error better than 10^{-5} for $x = \pi/2$

Trick:

$$\cos(t) = 1 - \frac{X^2}{2!} + \frac{X^4}{4!} - \cdots + (-1)^n \frac{X^{2n}}{(2n)!} + \underbrace{(-1)^{n+1}\frac{X^{2n+2}}{(2n+2)!}\cos(c_x)}_{\text{this is the Remainder.}}$$

let $t = x^2$

then
$$\cos(x^2) = 1 - \frac{X^4}{2!} + \frac{X^8}{4!} - \cdots + (-1)^n\frac{X^{4n}}{(2n)!} + (-1)^{n+1}\frac{X^{4n+4}}{(2n+2)!}\cos(c_x)$$

NOW BOUND THE REMAINDER...

$$R_{4n}(x) = (-1)^{n+1}\frac{X^{4n+4}}{(2n+2)!}\cos(C_n)$$ don't care about $\pm$ error

$$|R_{4n}| = \frac{X^{4n+4}}{(2n+2)!}\cos(C_n) \leq \frac{X^{4n+4}}{(2n+2)!}$$

and for our example $x = \pi/2$

$$|R_{4n}(\pi/2)| \leq \frac{(\pi/2)^{4n+4}}{(2n+2)!} < \frac{2^{4n+4}}{(2n+2)!}$$

could do this but it would way over estimate the # of terms needed

$n=5$ $\quad |R_{20}(\pi/2)| \leq 1.06\times10^{4}$

$n=6$ $\quad |R_{24}(\pi/2)| \leq 3.56\times10^{6}$ ✓

$$P_{24}(x) = 1 - \frac{X^4}{2!} + \frac{X^8}{4!} - \frac{X^{12}}{6!} + \frac{X^{16}}{8!} - \frac{X^{20}}{10!} + \frac{X^{24}}{12!} \sim \cos(x^2)$$

and $P_{24}(\pi/2) \sim \cos((\pi/2)^2)$

with error $< 10^{-5}$