Bring Rulers.

Numerical Analysis
Day 6.

<u>Last Time</u> Bisection method.

1 choose a starting range $[a, b]$
2. check $f(a)f(b) < 0$
3. bisect the region $c = (a+b)/2$ · This is our guess at the root.
4. choose which side $[a, c]$ or $[c, b]$ the root is on.
5. set the new range

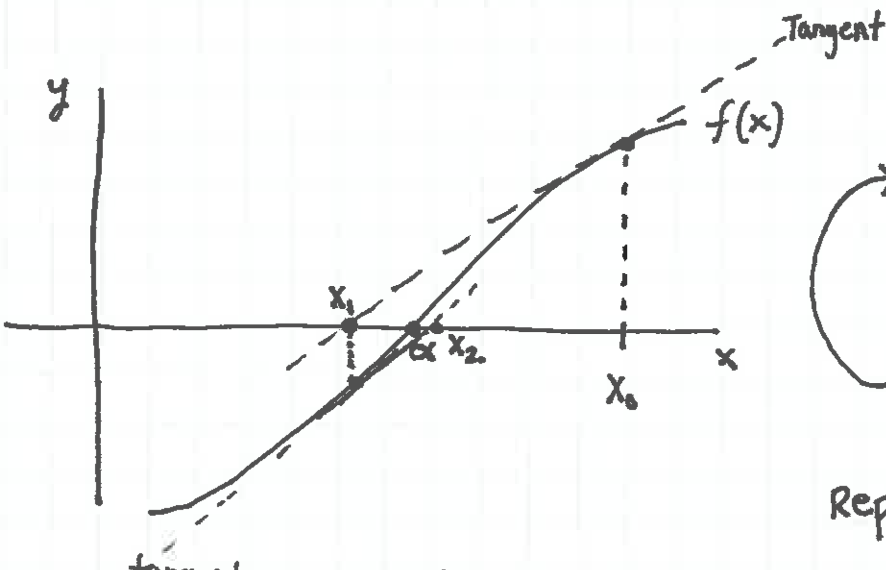
repeat

if $b - a < \epsilon$ then we stop.

Also know: Error $= |\alpha - c_n| \leq \frac{1}{2^n}(b-a)$

* This converges faster than just marching along the x-axis.
* We still have problems with FUZZY DATA.
  - zoom in too much we get $f(c)f(b) > 0$ and $f(a)f(c) > 0$ no root?

3.2 Newtons Method.



$f(\alpha) = 0$

1. Choose a starting point close to α. Call this x_0
2. Find the tangent line at $f(x_0)$
3. Look for the root of the tanget line. x_1
4. x_1 is our new guess at the root.

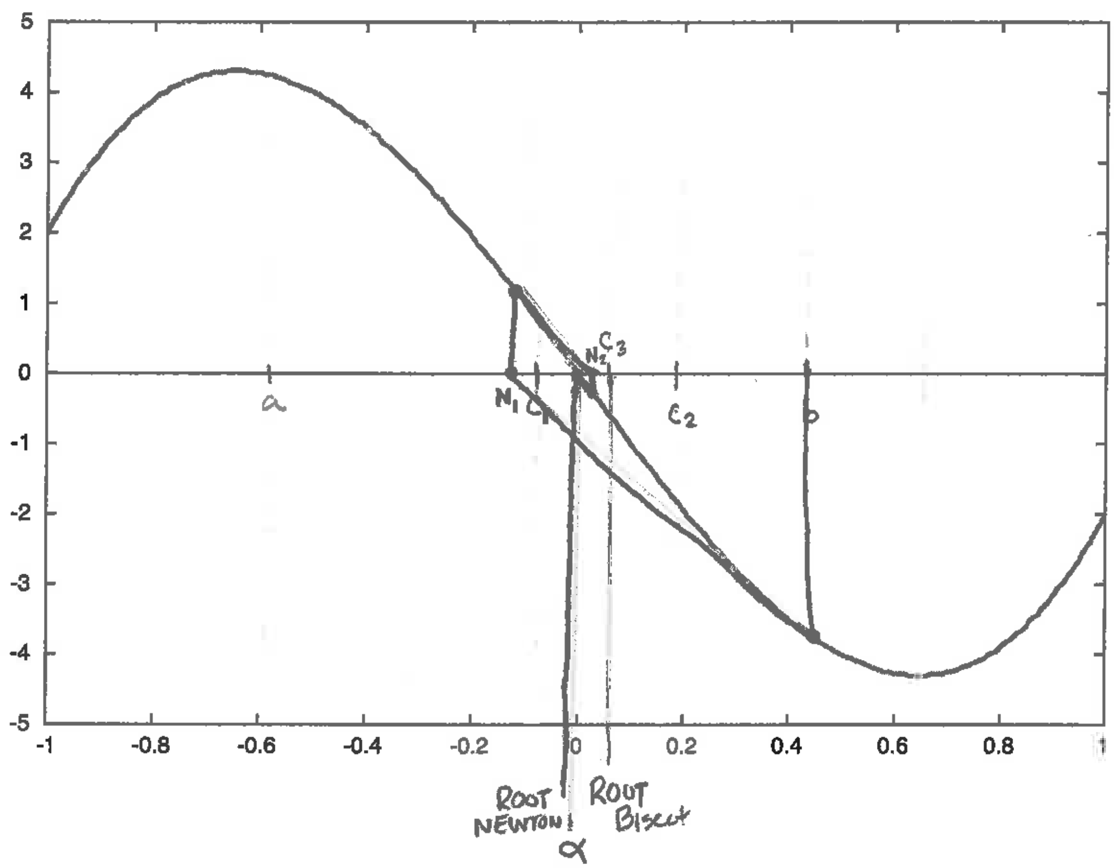
Repeat until you are within the error tollerance.

COMPARE... Bisection and Newtons Method.

$f(x) = 8x^3 - 10x$

Do three iterations of each method by hand.

What do you find?



Let's think about what we are doing Mathematically:

Given a function $f(x)$ with at least one real root at $f(\alpha)=0$.

1 Assume x_0 is sufficiently close to α. — on the picture could we pick a bad x_0?

2. Find the line tangent to $f(x)$ at x_0

$$\text{slope: } f'(x_0) \quad \text{point: } (x_0, f(x_0))$$

$$\text{Tangent Line} = y_1 = f(x_0) + f'(x_0)(x - x_0)$$

3. Find the root of the tangent line.

$$f(x_0) + f'(x_0)(x - x_0) = 0$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

4 call this x_1 our new guess at α

IN GENERAL: $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ This is NEWTON – RAPHSON METHOD

LAST TIME — Bisection Method

to find the root of $f(x) = x^6 - x - 1$

$\alpha \sim 1.134724138$

it took around $n=10$ interactions

to get $c_{10} = 1.338$ $\quad \epsilon = .001$

$\alpha - c_{10} = 9.24 \times 10^{4}$

YOU TRY — Newtons Method

1. Choose an x_0 try $x_0 = 1.5$
2. plug into $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Write your PSUEDO CODE.

$f(x_n) = x_n^6 - x_n - 1$
$f'(x_n) = 6x_n^5 - 1$

You calculate these by hand.

$$x_{n+1} = x_n - \frac{x_n^6 - x_n - 1}{6x_n^5 - 1}$$

How many iterations to do better than $\alpha - x_n = 9.24 \times 10^{4}$

An Interesting Application...

<u>Division</u> is computationally SLOW!
Think about doing Long Division by hand.

Computers use Newtons Method to speed things up.

SAY we want $a/b = a \cdot \left(\frac{1}{b}\right)$

multiplication is a faster operation.

We use a trick to find $\left(\frac{1}{b}\right)$

$b - \frac{1}{x} = 0$ the root of this is $1/b$.

plug $f(x) = b - \frac{1}{x}$ into NEWTONS METHOD.

$f(x_n) = b - 1/x_n \quad f'(x_n) = 1/x_n^2$

$$x_{n+1} = x_n - \frac{b - 1/x_n}{1/x_n^2} = x_n - bx_n^2 + x_n$$

or

$$x_{n+1} = x_n(2 - bx_n)$$

all easy calculations and fast convergence.

Only Problem... This could introduce ERROR but HOW MUCH?

<u>ERROR ANALYSIS</u> NEWTONS METHOD. — we need to understand the error.

Assume that $f'(\alpha) \neq 0$

we have to be careful in cases where $f(x)$ is tangent at the root!

Consider the taylor expansion of $f(x)$ near $x = x_n$ and use this to approximate $f(\alpha)$.

Newtons Method is a linear approximation.

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \underbrace{\tfrac{1}{2}(\alpha - x_n)^2 f''(c_x)}_{\text{Remainder}}$$

"Use what we know at $x = x_n$ to approximate $f(\alpha)$"

BUT we also know that $f(\alpha)=0$ ... we are looking for the root

$$0 = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 f''(c_x)$$

divide by $f'(x_n)$ because $(\alpha - x_n)$ is my error.

$$0 = \underbrace{\frac{f(x_n)}{f'(x_n)}}_{} + (\alpha - x_n) + \frac{1}{2}(\alpha - x_n)^2 \frac{f''(c_x)}{f'(x_n)}$$

replace with the method. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$0 = x_n - x_{n+1} + \alpha - x_n + \frac{1}{2}(\alpha - x_n)^2 \frac{f''(c_x)}{f'(x_n)}$$

x_n's cancel ... solve for $\alpha - x_n$

$$\alpha - x_{n+1} = (\alpha - x_n)^2 \left[\frac{-f''(c_x)}{2f'(x_n)}\right]$$

What do we learn ... error in the next step is proportional to the error in the previous step.

Convergence or decrease in error depends on the curvature and slope of $f(x)$ near $\alpha = x$.

TAKE A CLOSER LOOK:

$$-\frac{f''(c_x)}{2f'(x_n)} \approx -\frac{f''(\alpha)}{2f'(\alpha)} \equiv M$$

as long as x_n is close to α

then $\quad \alpha - x_{n+1} \approx M(\alpha - x_n)^2$

$$\text{OR} \quad M(\alpha - x_{n+1}) \approx [M(\alpha - x_n)]^2 \approx [[M(\alpha - x_{n-1})]^2]^2 \ldots$$

$$\text{so} \quad M(\alpha - x_{n+1}) \approx [M(\alpha - x_0)]^{2n}$$

What do we need for the method to converge or error to decrease.

We must have $|M(\alpha - x_0)| < 1$

otherwise error will grow.

Expand this a bit $|\alpha - x_0| < \frac{1}{|M|} = \left|\frac{2f'(\alpha)}{f''(\alpha)}\right|$

What did we learn...

- We have to pick x_0 sufficently close to α for the method to converge.
- How close depends on $f'(\alpha)$ and $f''(\alpha)$

* The smaller the slope $f'(\alpha)$ the better we need to guess x_0 for the method to converge.

ERROR ESTIMATION

- we can't actually use this definition of error in our code.
  - We don't know α.
  - Clunky to divide by $f''(\alpha)$ etc...

USE THIS IN CODE !!

We can estimate the error: $\boxed{\alpha - x_n \approx x_{n+1} - x_n}$

the error in the n^{th} step $\approx$ the difference between this step and the next.

MATHEMATICALLY:

subtract $f(\alpha)=0$

Mean Value THM.

$$f(x_n) = f(x_n) - f(\alpha) = f(x_n) - \underbrace{f(x_n) - (\alpha - x_n) f'(c_x)}_{\text{taylors thm for } f(\alpha)}$$

or

$$f(x_n) = f'(c_x)(x_n - \alpha)$$

$$(\alpha - x_n) = -\frac{f(x_n)}{f'(c_x)} \approx -\frac{f(x_n)}{f'(x_n)}$$

assuming x_n is close to α

c_x is between x_n and α.

BUT newtons method $x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$

so $(\alpha - x_n) \approx (x_{n+1} - x_n)$