Last week: BISECTION, NEWTONS > How did the programming go?

SKIP – secant method.

EX How do we apply newtons method to find the roots of

$$f(x) = x^2 - 5 \qquad f(\alpha) = 0 \quad \alpha = \pm\sqrt{5}$$

need $f'(x) = 2x$

Newton $$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}$$

start with an initial guess, x_0, and hope our solution converges to the root.

IN GENERAL... this is called FIXED POINT ITERATION.

The roots are fixed points... if we plug in $x_n = \sqrt{5}$

$$x_{n+1} = \sqrt{5} - \frac{(\sqrt{5})^2 - 5}{2\sqrt{5}} = \sqrt{5}$$

we get the same point back.

FIXED POINT – a point that is mapped to itself by the function.

We could study the general class of methods:

$$x_{n+1} = g(x_n) \quad \text{such That} \quad \alpha = g(\alpha)$$

For example [WORKSHEET]

$$\left.\begin{aligned} x_{n+1} &= 5 + x_n - x_n^2 \\ x_{n+1} &= 5/x_n \\ x_{n+1} &= 1 + x_n - \tfrac{1}{5}x_n^2 \\ x_{n+1} &= \tfrac{1}{2}(x_n + 5/x_n) \end{aligned}\right\}$$

NEWTON.

all of these have $\alpha = \sqrt{5}$ is a fixedpoint.

why not just pick the most simple iteration that gets me to my root.

What did we see ...
1. all eqns have fixed points $\alpha = \sqrt{5}$
2. NOT all equations converge to the fixed point.

The Question: How can we be sure that our iterative method has a unique soln and that it will converge to that soln for a given initial guess?

<u>GOALS</u> Given $\bar{x} = g(\bar{x})$ or $x_{n+1} = g(x_n)$

1. When will solutions exist?
2. If a solution exists is it unique?
3. Will our iteration converge to the unique soln?

We are going to investigate and prove some ideas in class today ... this will lead us to a better understanding of our methods!

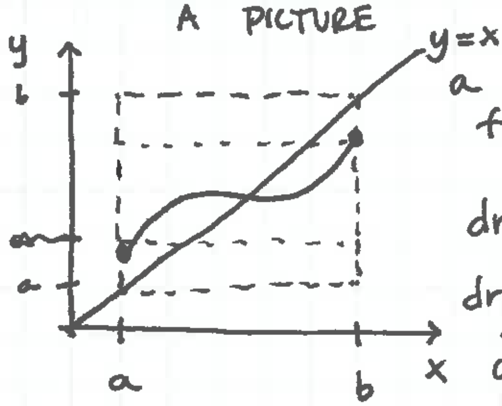
Existance of Solns $\longrightarrow$ Lemma 3.4.1

Let $g(x)$ be continuous on $[a, b]$ and suppose it satisfies

$$a \le x \le b \Rightarrow a \le g(x) \le b$$

then the equation $x = g(x)$ has at least one solution on the interval $[a, b]$.

FIRST DRAW A PICTURE



a and b frame a window.

draw $y = x$

draw $g(x)$ that starts above a and below b.

The two MUST CROSS!!

PROOF:

Assume $g(x)$ is continuous and define $f(x) = x - g(x)$. This is continuous on $[a, b]$ and if $a \le x \le b \Rightarrow a \le g(x) \le b$ then $f(a) \le 0$ and $f(b) \ge 0$. Thus the intermediate value theorem implies there must be a point $x \in [a, b]$ for which $f(x) = 0$ and $x = g(x)$ $\square$

So as long as we choose a good viewing window $a \le x \le b$ a solution should exist.

UNIQUENESS and CONVERGENCE of Solns $\longrightarrow$ Theorem 3.4.2

The Contraction Mapping Theorem. (CMT) $x_{n+1} = g(x_n)$

Assume $g(x)$ and $g'(x)$ are continuous on $a \le x \le b$ and that $a \le x \le b \Rightarrow a \le g(x) \le b$. Further assume that

$$\lambda \equiv \max_{a \le x \le b} |g'(x)| < 1$$

then

1. There is a unique solution α of $x = g(x)$ in $[a,b]$
2. For any initial estimate $x_0 \in [a,b]$ the iterates x_n will converge to α
3. The error of the n^{th} iteration is bounded by
$$|\alpha - x_n| \le \frac{\lambda^n}{1-\lambda} |x_0 - x_1|$$
4. $$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$$ thus for x_n close to α
$$\alpha - x_{n+1} \approx g'(\alpha)(\alpha - x_n)$$

WOW! We just got all our wishes granted!!

Let's consider the proof of each of these and see what we learn.

RECALL Mean Value Theorem — Let $f(x)$ be continuous for $a \le x \le b$ and differentiable for $a < x < b$. Then there exists at least one point $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a)$$

So $$|f(b) - f(a)| = |f'(c)||b-a| < \lambda |b-a|$$

as defined in CMT.

PROOF 1. UNIQUENESS — how do we prove uniqueness? Assume two contradiction show equal.

Assume $\alpha = g(\alpha)$ and $\beta = g(\beta)$ for $\alpha, \beta \in (a,b)$

then

$$|\alpha - \beta| \underset{\text{assumed in CTM}}{\leq} |g(\alpha) - g(\beta)| \underset{\text{Mean Valve THM}}{=} |g'(c)||\alpha - \beta| \underset{\text{Def. of } \lambda.}{\leq} \lambda |\alpha - \beta|$$

so $(1-\lambda)|\alpha - \beta| \leq 0$

either $(1-\lambda) \leq 0$ and $\lambda \geq 1$ contradicting the theorem

or $|\alpha - \beta| = 0$ and $\alpha = \beta$ the soln is unique. □

PROOF 2. CONVERGENCE — use the definition of error... convergence means error gets smaller with each iteration.

Assume that $\alpha = g(\alpha)$ and consider $x_{n+1} = g(x_n)$

then $\underbrace{\alpha - x_{n+1}}_{\text{THE ERROR}} \underset{\text{plug in the method}}{=} g(\alpha) - g(x_n) \underset{\text{Mean Value THM.}}{=} g'(c_n)(\alpha - x_n)$

for some c_n between α and x_n

thus $|\alpha - x_{n+1}| = |g'(c_n)||\alpha - x_n| \leq \lambda |\alpha - x_n|$

definition of $\lambda \equiv \max|g'(x)|$

but by induction

$$|\alpha - x_{n+1}| \leq \lambda |\alpha - x_n| \leq \lambda^2 |\alpha - x_{n-1}| \leq \lambda^n |\alpha - x_0|$$

NOTE ... what term seems to be REALLY IMPORTANT? $\lambda < 1$!!!

because we require $\lambda < 1$ as $n \to \infty$ $|\alpha - x_{n+1}| \to 0$ and x_{n+1} converges to α.

□

PROOF 3. BOUNDS ON THE ERROR

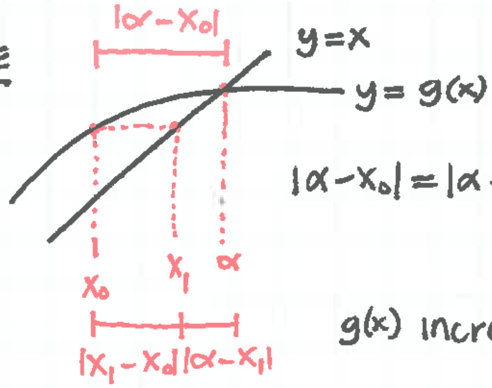
— consider the error in the initial guess and relate this to the error in the n^{th} guess.

Assume our method is converging. Then

$$|\alpha - x_0| \le |\alpha - x_1| + |x_1 - x_0|$$

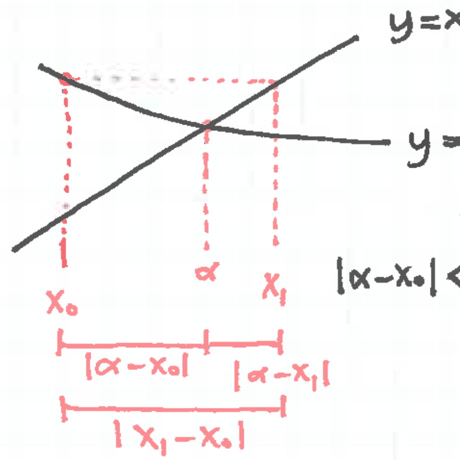
DRAW A PICTURE.
COBWEB DIAGRAM.

ASIDE



$|\alpha - x_0| = |\alpha - x_1| + |x_1 - x_0|$

$g(x)$ increasing



$g(x)$ decreasing

$|\alpha - x_0| < |\alpha - x_1| + |x_1 - x_0|$

$$|\alpha - x_0| \le |\alpha - x_1| + |x_1 - x_0| \le \lambda|\alpha - x_0| + |x_1 - x_0|$$

MVT and def of λ
We showed this in PROOF 2.

so $\quad (1-\lambda)|\alpha - x_0| \le |x_1 - x_0| \quad$ gather like terms

and $\quad |\alpha - x_0| \le \dfrac{1}{(1-\lambda)}|x_1 - x_0|$

solve for the error.

but $\quad |\alpha - x_1| \le \lambda|\alpha - x_0| \le \dfrac{\lambda}{(1-\lambda)}|x_1 - x_0|$

PROVED THIS In PROOF 2.

and by induction

$$|\alpha - x_n| \le \frac{\lambda^n}{(1-\lambda)}|x_1 - x_0| \quad \blacksquare$$

Bounds the error in our n^{th} step.

PROOF 4. ERROR ESTIMATION — as we get close to the soln, x_n near α, how quickly will our error decrease.

Consider the error of the $n+1^{th}$ step.

$$\alpha - x_{n+1} = g(\alpha) - g(x_n) = g'(c_n)(\alpha - x_n)$$

plug into Method

MEAN VALUE.

then

$$\frac{(\alpha - x_{n+1})}{(\alpha - x_n)} = g'(c_n)$$

as we get very close to the true solution

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$$

Because c_n is between x_n and α and as $n \to \infty$ $x_n \to \infty$ so all three points must merge.

For large n we write

$$\alpha - x_{n+1} \approx g'(\alpha)(\alpha - x_n) \quad \square$$

When we are very close to the root our method will converge at a constant rate of $g'(\alpha)$.
Speed of convergence depends on the size of $g'(\alpha)$.

LOOK BACK AT THE EXAMPLE...

$$x_{n+1} = 5 + x_n - x_n^2 \qquad \text{here} \quad g(x_n) = 5 + x_n - x_n^2$$

HOW DO WE USE CONTRACTION MAPPING.

$$\text{so} \quad g'(x_n) = 1 - 2x_n$$

near our root $\alpha = \sqrt{5}$

$$g'(\sqrt{5}) = 1 - 2\sqrt{5} \approx -3.472.$$

This iteration will not converge!

Even though we have a unique soln the iterations will never get us there.

TEST the other functions on the worksheet.

• NEXT TIME AITKIN EXTRAPOLATION.