DAY 9
NUMERICAL ANALYSIS

problem 12

From the Homework.

solve for fixed points: $\bar{x} = g(\bar{x})$

$\bar{x} = c\bar{x}(1-\bar{x}) = c\bar{x} - c\bar{x}^2$

$c\bar{x}^2 + (1-c)\bar{x} = 0$

$\bar{x}[c\bar{x} + (1-c)] = 0$

$\bar{x} = 0 \qquad c\bar{x} = -\frac{(1-c)}{c}$

$\bar{x} = \frac{c-1}{c}$

TWO FIXED POINTS $\quad \bar{x} = 0$ ✓ and $\bar{x} = 1 - \frac{1}{c}$ ✓

convergence depends on $\lambda \equiv \max_{a<x<b} |g'(x)| < 1$ From the Contraction Mapping Theorem.

consider $g'(x)$: $\quad g(x) = cx(1-x)$

$$g'(x) = c(1-x) + cx(-1) = c - cx - cx = c(1-2x)$$

need $|c||1-2x| < 1$ to converge.

If we want $\alpha = 1$ then $|c| < 1$ should work.

$\boxed{-1 < c < 1}$

Changing the value of c changes which root you converge to 0 or $1-\frac{1}{c}$

If we want $\alpha = 1 - \frac{1}{c}$ then

$|c(1-2(1-\frac{1}{c}))| = |c - 2c + 2|$

$= |2 - c| < 1$

$-1 < 2 - c < 1$

$-3 < -c < -1$

$\boxed{1 < c < 3}$ should work.

- Outside of these ranges the iteration diverges!

FreeMat:

logistictest(.5, 100, c)

look what happens.

logistictest (xzero, 100, 2) → Desmos ... plot $y = x$, $y = cx(1-x)$

you can make a bad guess $0 < x_{zero} < 1$.

This diverges even though c is within range.

More: Evil and Diabolical Problems:

Issues with Stability.

We expect life to be STABLE ... small changes in one area $\longrightarrow$ small changes in another.

Ex If I cut .001 Calories per day how much weight will I loose tomorrow?

There are some cases where this does not happen!

ILL-CONDITIONED or UNSTABLE problems.
A small change in the initial conditions or parameters completely changes the results.
This often arises from the modeling process.

Ex Consider $f(x) = x^2 - 2x + 1 = (x-1)(x-1)$
double root at $x=1$

What are some issues w/ multiple roots?

Q: What if in the modeling process we made an error in measuring $b=-2$. What if the "real" constant should have been $b=-2.0001$

An error of $E = .0001$ in measurement

Freemat:
illcondition(-.0001) new roots $x_1 = 1.010050$
$x_2 = 0.990050$

The error in our solution was magnified $E \sim .01$

The small error in measurement leads to a much larger error in our final solution.

In fact if $b=-1.9999$ our roots become complex #'s.
Completly changing the soln.

It's good to know a-priori if your model is ill conditioned or unstable!

PERTURBATION ANALYSIS — goal — "push" or perturb the function a bit to see what happens.

For the root finding problem, consider

$$F_\epsilon = f(x) + \epsilon g(x)$$

For our example this is $F_\epsilon = (x^2 - 2x + 1) + \epsilon x$

$\epsilon << 1$ is a small change. we let $\epsilon = -.0001$

The roots of F_ϵ will depend on ϵ so we write

$$F(\alpha(\epsilon)) = 0$$

$\alpha(\epsilon)$ is the root and $\alpha(0)$ is the root of the original problem

In the example $\alpha(-.0001) = 1.010050$
$\alpha(0) = 1$

Let's find an expression to approximate $\alpha(\epsilon)$ for ϵ close to zero.

hmm... does this sound like Taylor series.

(★) $\alpha(\epsilon) \sim \alpha(0) + \epsilon\alpha'(0) + \text{HOT}$ Higher order Terms.

Now lets see how this change in $\alpha(0) \to \alpha(\epsilon)$ looks in our original function. GOAL — sub in for $\alpha'(0)$

$$F_\epsilon(\alpha(\epsilon)) = f(\alpha(\epsilon)) + \epsilon g(\alpha(\epsilon)) = 0$$

$\alpha(\epsilon)$ is a root.

Take the derivative (wrt ϵ) "how does this ϵ cause change"

$$f'(\alpha(\epsilon))\alpha'(\epsilon) + g(\alpha(\epsilon)) + \epsilon g'(\alpha(\epsilon))\alpha'(\epsilon) = 0$$

if $\epsilon = 0$ then

$$f'(\alpha(0))\alpha'(0) + g(\alpha(0)) = 0$$

and

$$\alpha'(0) = -\frac{g(\alpha(0))}{f'(\alpha(0))}$$

plug this into the approx (★)

$$\boxed{\alpha(\epsilon) \sim \alpha(0) - \epsilon \frac{g(\alpha(0))}{f'(\alpha(0))}}$$

this will let us predict how a small change effects the root.

Look back at the example...

Aside

$f(x) = x^2 - 2x + 1$

Multiple roots always give us trouble!!

$g(x) = \epsilon x$

how does changing the coeff in front of x change the final root.

$f'(x) = 2x - 2$

Here $\alpha(0) = 1$

so $f'(\alpha(0)) = f'(1) = 0$ $\qquad g(1) = \epsilon$

PROBLEM — with repeated roots we have to be careful we actually need to keep more higher order terms. (More advanced Perturbation analysis)

EX ONE THAT WORKS...

$$f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)$$
$$= x^7 - 28x^6 + 322x^5 - 1960x^4 + 6769x^3 - 13132x^2 + 13068x - 5040$$

If we solve on a computer we find.

Consider the root $x=4$
If we change $-28x^6$ to $-28.002x^6$ we find the new root is $x = 3.81957$

A measurement error of .002 leads to a result error of .180 PROBLEM!!

Use Perturbation Analysis:

$F_\epsilon = f(x) + \epsilon x^6$ $\qquad$ we are changing the x^6 coefficient.

$f'(x) =$ a bunch of stuff that goes to zero when I plug in $x=4$ $+ (x-1)(x-2)(x-3)(x-5)(x-6)(x-7)$

$$f'(4) = -36 \quad \text{and} \quad g(4) = 4^6 = 4096$$

$$\alpha(\epsilon) \sim 4 - \epsilon \frac{4096}{-36} \sim 4 + 114\epsilon$$

small ϵ has a big effect on the root!

Interesting note: $\alpha(-.002) \sim 4 + 114(-.002) = 3.772$
this approximates our root for $-28.002x^6$.

YOU TRY — What happens to the root $x=3$ when we change from $-28x^6$ to $-28.002x^6$

$\alpha(\epsilon) \approx 3 - 15.2\,\epsilon$
so $\alpha(.002) \approx 3.0304$

the effects are not equally drastic on all roots.

OTHER INTERESTING ISSUES...
CHAOS in NEWTONS METHOD.

Chaos — a small change in the initial condition (OR INITIAL GUESS x_0) causes a huge change in the solution.

"The present always determines the future but the approximate present does not approximately determine the future."

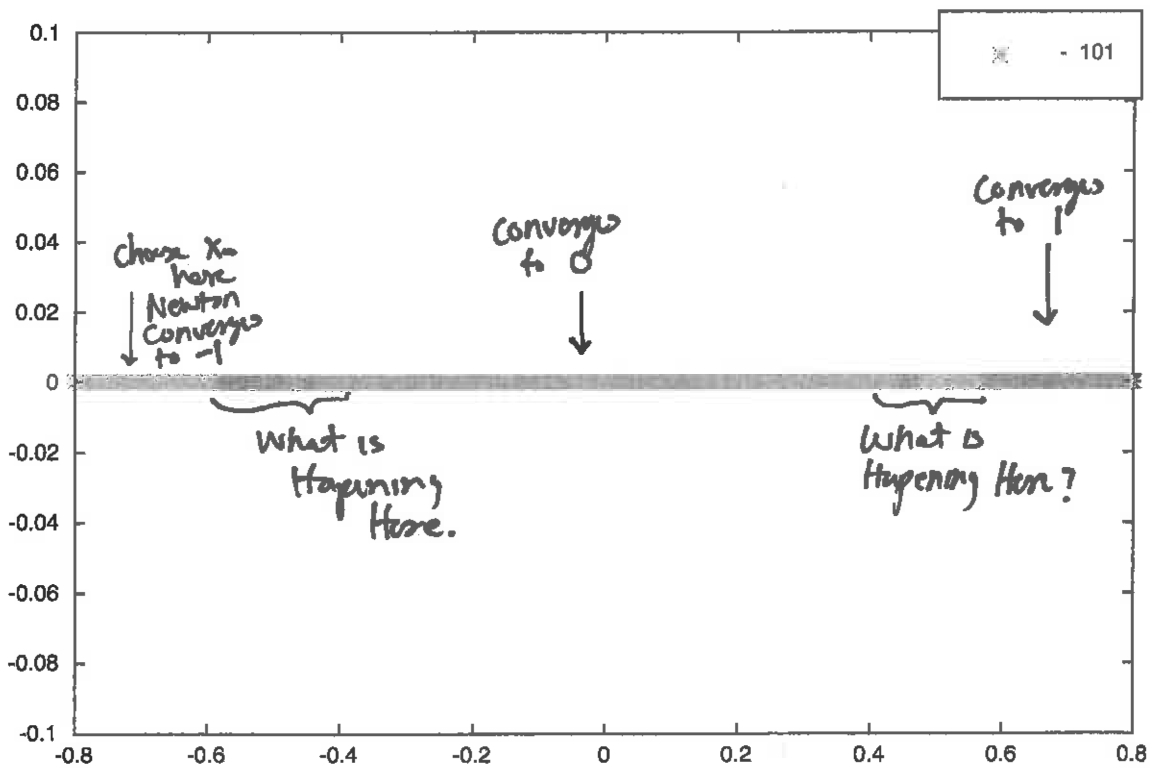
Ex $f(x) = x^3 - x$ has roots $x = -1, 0, 1$
Use newtons method to solve for these roots.

CODE... Lets choose starting values from between $-.8$ and $.8$, $.4$ and $.6$, keep zooming in.

RED — point converges to -1
GREEN — point converges to 0
BLUE — point converges to 1

newton_fractal(-.8, .8, 1000, .001)
(.4, .6, 1000, .001)
(.445, .455, 1000, .001)

zoom in you keep seeing more and more regions of switching between -1 and 1.



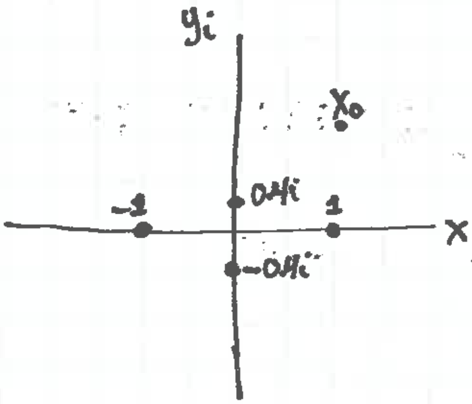
- 101
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
Chose x_0 here Newton Converges to -1
Converges to 0
Converges to 1
What is Happening Here.
What is Happening Here?

An interesting and beautiful thing happens if we extend newtons method to the Complex plane.

Define $f(z) = (z-1)(z+1)(z-0.4i)(z+0.4i)$

Roots $z = 1, -1, 0.4i, -0.4i$

We can use our newtons method on this choosing starting points from any where in the complex plane.



Choose an X_0... which root does it converge to?

Converge to 1 — color yellow
-1 — color red
0.4i — color green
-0.4i — color blue
Wasteland = No Color

Try newtonpic(400, [-.15, .55], [-.2, .2])

Make an NxN grid — 400
xrange — [-.15, .55]
yrange — [-.2, .2]

newtonzoom? Very Slow!

Thanks to Computer Engineering at RICE university for preparing this m-code.