- if $r > 1$, then $\sum a_n$ diverges
- if $r = 1$, then $\sum a_n$ could converge or diverge.

(This test works since $\lim_{n\to\infty} \sqrt[n]{a_n} = r$ tells us that the series is comparable to a geometric series with ratio r.) Use this test to determine the behavior of the series.

94. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^n$

95. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5n+1}{3n^2}\right)^n$

Strengthen Your Understanding

In Problems 96–98, explain what is wrong with the statement.

96. The series $\sum(-1)^{2n}/n^2$ converges by the alternating series test.

97. The series $\sum 1/(n^2+1)$ converges by the ratio test.

98. The series $\sum 1/n^{3/2}$ converges by comparison with $\sum 1/n^2$.

In Problems 99–101, give an example of:

99. A series $\sum_{n=1}^{\infty} a_n$ that converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

100. An alternating series that does not converge.

101. A series $\sum a_n$ such that

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 3.$$

Decide if the statements in Problems 102–117 are true or false. Give an explanation for your answer.

102. If the terms s_n of a sequence alternate in sign, then the sequence converges.

103. If $0 \le a_n \le b_n$ for all n and $\sum a_n$ converges, then $\sum b_n$ converges.

104. If $0 \le a_n \le b_n$ for all n and $\sum a_n$ diverges, then $\sum b_n$ diverges.

105. If $b_n \le a_n \le 0$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges.

106. If $\sum a_n$ converges, then $\sum |a_n|$ converges.

107. If $\sum a_n$ converges, then $\lim\limits_{n\to\infty} |a_{n+1}|/|a_n| \neq 1$.

108. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\cos(2\pi n)$ is an alternating series.

109. The series $\displaystyle\sum_{n=0}^{\infty}(-1)^n 2^n$ converges.

110. If $\sum a_n$ converges, then $\sum(-1)^n a_n$ converges.

111. If an alternating series converges by the alternating series test, then the error in using the first n terms of the series to approximate the entire series is less in magnitude than the first term omitted.

112. If an alternating series converges, then the error in using the first n terms of the series to approximate the entire series is less in magnitude than the first term omitted.

113. If $\sum |a_n|$ converges, then $\sum(-1)^n|a_n|$ converges.

114. To find the sum of the alternating harmonic series $\sum(-1)^{n-1}/n$ to within 0.01 of the true value, we can sum the first 100 terms.

115. If $\sum a_n$ is absolutely convergent, then it is convergent.

116. If $\sum a_n$ is conditionally convergent, then it is absolutely convergent.

117. If $a_n > 0.5b_n > 0$ for all n and $\sum b_n$ diverges, then $\sum a_n$ diverges.

118. Which test will help you determine if the series converges or diverges?

$$\sum_{k=1}^{\infty}\frac{1}{k^3+1}$$

(a) Integral test
(b) Comparison test
(c) Ratio test

9.5 POWER SERIES AND INTERVAL OF CONVERGENCE

In Section 9.2 we saw that the geometric series $\sum ax^n$ converges for $-1 < x < 1$ and diverges otherwise. This section studies the convergence of more general series constructed from powers. Chapter 10 shows how such power series are used to approximate functions such as e^x, $\sin x$, $\cos x$, and $\ln x$.

A **power series about $x = a$** is a sum of constants times powers of $(x-a)$:

$$C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n + \cdots = \sum_{n=0}^{\infty} C_n(x-a)^n.$$

We think of a as a constant. For any fixed x, the power series $\sum C_n(x-a)^n$ is a series of numbers like those considered in Section 9.3. To investigate the convergence of a power series, we consider the partial sums, which in this case are the polynomials $S_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n$. As before, we consider the sequence[8]

$$S_0(x),\ S_1(x),\ S_2(x),\ \ldots,\ S_n(x),\ldots.$$

For a fixed value of x, if this sequence of partial sums converges to a limit S, that is, if $\lim_{n\to\infty} S_n(x) = S$, then we say that the power series **converges** to S for this value of x.

A power series may converge for some values of x and not for others.

Example 1 Find an expression for the general term of the series and use it to write the series using $\sum$ notation:

$$\frac{(x-2)^4}{4} - \frac{(x-2)^6}{9} + \frac{(x-2)^8}{16} - \frac{(x-2)^{10}}{25} + \cdots.$$

Solution The series is about $x = 2$ and the odd terms are zero. We use $(x-2)^{2n}$ and begin with $n = 2$. Since the series alternates and is positive for $n = 2$, we multiply by $(-1)^n$. For $n = 2$, we divide by 4, for $n = 3$ we divide by 9, and in general, we divide by n^2. One way to write this series is

$$\sum_{n=2}^{\infty} \frac{(-1)^n (x-2)^{2n}}{n^2}.$$

Example 2 Determine whether the power series $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converges or diverges for

(a) $x = -1$

(b) $x = 3$

Solution (a) Substituting $x = -1$, we have

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n.$$

This is a geometric series with ratio $-1/2$, so the series converges to $1/(1-(-\frac{1}{2})) = 2/3$.

(b) Substituting $x = 3$, we have

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{3^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n.$$

This is a geometric series with ratio greater than 1, so it diverges.

Numerical and Graphical View of Convergence

Consider the series

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots.$$

To investigate the convergence of this series, we look at the sequence of partial sums graphed in

[8] Here we call the first term in the sequence $S_0(x)$ rather than $S_1(x)$ so that the last term of $S_n(x)$ is $C_n(x-a)^n$.

Figure 9.11. The graph suggests that the partial sums converge for x in the interval $(0, 2)$. In Examples 4 and 5, we show that the series converges for $0 < x \leq 2$. This is called the *interval of convergence* of this series.

At $x = 1.4$, which is inside the interval, the series appears to converge quite rapidly:

$$S_5(1.4) = 0.33698\ldots \quad S_7(1.4) = 0.33653\ldots$$
$$S_6(1.4) = 0.33630\ldots \quad S_8(1.4) = 0.33645\ldots$$

Table 9.1 shows the results of using $x = 1.9$ and $x = 2.3$ in the power series. For $x = 1.9$, which is inside the interval of convergence but close to an endpoint, the series converges, though rather slowly. For $x = 2.3$, which is outside the interval of convergence, the series diverges: the larger the value of n, the more wildly the series oscillates. In fact, the contribution of the twenty-fifth term is about 28; the contribution of the hundredth term is about $-2{,}500{,}000{,}000$. Figure 9.11 shows the interval of convergence and the partial sums.

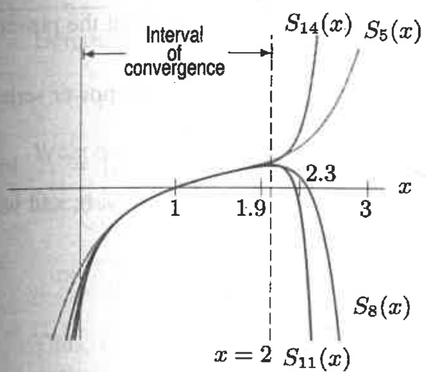

Figure 9.11: Partial sums for series in Example 4 converge for $0 < x < 2$

Table 9.1 *Partial sums for series in Example 4 with $x = 1.9$ inside interval of convergence and $x = 2.3$ outside*

n	$S_n(1.9)$	n	$S_n(2.3)$
2	0.495	2	0.455
5	0.69207	5	1.21589
8	0.61802	8	0.28817
11	0.65473	11	1.71710
14	0.63440	14	−0.70701

Notice that the interval of convergence, $0 < x \leq 2$, is centered on $x = 1$. Since the interval extends one unit on either side, we say the *radius of convergence* of this series is 1.

Intervals of Convergence

Each power series falls into one of three cases, characterized by its *radius of convergence, R.* This radius gives an *interval of convergence.*

- The series converges only for $x = a$; the **radius of convergence** is defined to be $R = 0$.
- The series converges for all values of x; the **radius of convergence** is defined to be $R = \infty$.
- There is a positive number R, called the **radius of convergence,** such that the series converges for $|x - a| < R$ and diverges for $|x - a| > R$. See Figure 9.12.

Using the radius of convergence, we make the following definition:

- The **interval of convergence** is the interval between $a - R$ and $a + R$, including any endpoint where the series converges.

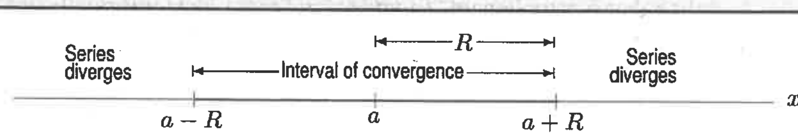

Figure 9.12: Radius of convergence, R, determines an interval, centered at $x = a$, in which the series converges

There are some series whose radius of convergence we already know. For example, the geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

converges for $|x| < 1$ and diverges for $|x| \geq 1$, so its radius of convergence is 1. Similarly, the series

$$1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \cdots + \left(\frac{x}{3}\right)^n + \cdots$$

converges for $|x/3| < 1$ and diverges for $|x/3| \geq 1$, so its radius of convergence is 3.

The next theorem gives a method of computing the radius of convergence for many series. To find the values of x for which the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ converges, we use the ratio test. Writing $a_n = C_n(x-a)^n$ and assuming $C_n \neq 0$ and $x \neq a$, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|C_{n+1}(x-a)^{n+1}|}{|C_n(x-a)^n|} = \lim_{n\to\infty} \frac{|C_{n+1}||x-a|}{|C_n|} = |x-a| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}.$$

Case 1. Suppose $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ is infinite. Then the ratio test shows that the power series converges only for $x = a$. The radius of convergence is $R = 0$.
Case 2. Suppose $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 0$. Then the ratio test shows that the power series converges for all x. The radius of convergence is $R = \infty$.
Case 3. Suppose $\lim_{n\to\infty} |a_{n+1}|/|a_n| = K|x-a|$, where $\lim_{n\to\infty} |C_{n+1}|/|C_n| = K$. In Case 1, K does not exist; in Case 2, $K = 0$. Thus, we can assume K exists and $K \neq 0$, and we can define $R = 1/K$. Then we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = K|x-a| = \frac{|x-a|}{R},$$

so the ratio test tells us that the power series:

- Converges for $\dfrac{|x-a|}{R} < 1$; that is, for $|x-a| < R$
- Diverges for $\dfrac{|x-a|}{R} > 1$; that is, for $|x-a| > R$.

The results are summarized in the following theorem.

Theorem 9.10: Method for Computing Radius of Convergence

To calculate the radius of convergence, R, for the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$, use the ratio test with $a_n = C_n(x-a)^n$.

- If $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ is infinite, then $R = 0$.
- If $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 0$, then $R = \infty$.
- If $\lim_{n\to\infty} |a_{n+1}|/|a_n| = K|x-a|$, where K is finite and nonzero, then $R = 1/K$.

Note that the ratio test does not tell us anything if $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ fails to exist, which can occur, for example, if some of the C_ns are zero.

A proof that a power series has a radius of convergence and of Theorem 9.10 is given in the online theory supplement. To understand these facts informally, we can think of a power series as being like a geometric series whose coefficients vary from term to term. The radius of convergence depends on the behavior of the coefficients: if there are constants C and K such that for larger and larger n,

$$|C_n| \approx CK^n,$$

then it is plausible that $\sum C_n x^n$ and $\sum CK^n x^n = \sum C(Kx)^n$ converge or diverge together. The geometric series $\sum C(Kx)^n$ converges for $|Kx| < 1$, that is, for $|x| < 1/K$. We can find K using the ratio test, because

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|C_{n+1}||(x-a)^{n+1}|}{|C_n||(x-a)^n|} \approx \frac{CK^{n+1}|(x-a)^{n+1}|}{CK^n|(x-a)^n|} = K|x-a|.$$

Example 3 Show that the following power series converges for all x:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Solution Because $C_n = 1/n!$, none of the C_ns are zero and we can use the ratio test:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x| \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = |x| \lim_{n\to\infty} \frac{n!}{(n+1)!} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

This gives $R = \infty$, so the series converges for all x. We see in Chapter 10 that it converges to e^x.

Example 4 Determine the radius of convergence of the series

$$\frac{(x-1)}{3} - \frac{(x-1)^2}{2\cdot 3^2} + \frac{(x-1)^3}{3\cdot 3^3} - \frac{(x-1)^4}{4\cdot 3^4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n\cdot 3^n} + \cdots.$$

What does this tell us about the interval of convergence of this series?

Solution Because $C_n = (-1)^{n-1}/(n\cdot 3^n)$ is never zero we can use the ratio test. We have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x-1| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x-1| \lim_{n\to\infty} \frac{\left|\frac{(-1)^n}{(n+1)\cdot 3^{n+1}}\right|}{\left|\frac{(-1)^{n-1}}{n\cdot 3^n}\right|} = |x-1| \lim_{n\to\infty} \frac{n}{3(n+1)} = \frac{|x-1|}{3}.$$

Thus, $K = 1/3$ in Theorem 9.10, so the radius of convergence is $R = 1/K = 3$. The power series converges for $|x-1| < 3$ and diverges for $|x-1| > 3$, so the series converges for $-2 < x < 4$. Notice that the radius of convergence does not tell us what happens at the endpoints, $x = -2$ and $x = 4$. The endpoints are investigated in Example 5.

What Happens at the Endpoints of the Interval of Convergence?

The ratio test does not tell us whether a power series converges at the endpoints of its interval of convergence, $x = a \pm R$. There is no simple theorem that answers this question. Since substituting $x = a \pm R$ converts the power series to a series of numbers, the tests in Sections 9.3 and 9.4 are often useful.

Example 5 Determine the interval of convergence of the series

$$\frac{(x-1)}{3} - \frac{(x-1)^2}{2\cdot 3^2} + \frac{(x-1)^3}{3\cdot 3^3} - \frac{(x-1)^4}{4\cdot 3^4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n\cdot 3^n} + \cdots.$$

Solution In Example 4 we showed that this series has $R = 3$; it converges for $-2 < x < 4$ and diverges for $x < -2$ or $x > 4$. We need to determine whether it converges at the endpoints of the interval of convergence, $x = -2$ and $x = 4$. At $x = 4$, we have the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots.$$

This is an alternating series with $a_n = 1/(n+1)$, so by the alternating series test (Theorem 9.8), it converges. At $x = -2$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots.$$

This is the negative of the harmonic series, so it diverges. Therefore, the interval of convergence is $-2 < x \leq 4$. The right endpoint is included and the left endpoint is not.

Series with All Odd, or All Even, Terms

The ratio test requires $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ to exist for $a_n = C_n(x-a)^n$. What happens if the power series has only even or odd powers, so some of the coefficients C_n are zero? Then we use the fact that an infinite series can be written in several ways and pick one in which the terms are nonzero.

Example 6 Find the radius and interval of convergence of the series

$$1 + 2^2x^2 + 2^4x^4 + 2^6x^6 + \cdots.$$

Solution If we take $a_n = 2^n x^n$ for n even and $a_n = 0$ for n odd, $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ does not exist. Therefore, for this series we take

$$a_n = 2^{2n}x^{2n},$$

so that, replacing n by $n+1$, we have

$$a_{n+1} = 2^{2(n+1)}x^{2(n+1)} = 2^{2n+2}x^{2n+2}.$$

Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|2^{2n+2}x^{2n+2}|}{|2^{2n}x^{2n}|} = |2^2x^2| = 4x^2.$$

We have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 4x^2.$$

The ratio test guarantees that the power series converges if $4x^2 < 1$, that is, if $|x| < \frac{1}{2}$. The radius of convergence is $\frac{1}{2}$. The series converges for $-\frac{1}{2} < x < \frac{1}{2}$ and diverges for $x > \frac{1}{2}$ or $x < -\frac{1}{2}$. At $x = \pm\frac{1}{2}$, all the terms in the series are 1, so the series diverges (by Theorem 9.2, Property 3). Thus, the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$.

Example 7 Write the general term a_n of the following series so that none of the terms are zero:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots.$$

Solution This series has only odd powers. We can get odd integers using $2n-1$ for $n \ge 1$, since

$$2\cdot 1 - 1 = 1, \qquad 2\cdot 2 - 1 = 3, \qquad 2\cdot 3 - 1 = 5, \text{ etc.}$$

Also, the signs of the terms alternate, with the first (that is, $n = 1$) term positive, so we include a factor of $(-1)^{n-1}$. Thus we get

$$a_n = (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}.$$

We see in Chapter 10 that the series converges to $\sin x$. Exercise 24 shows that the radius of convergence of this series is infinite, so that it converges for all values of x.