SOLUTIONS

Differential Equations - Nonlinear Systems and Applications

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Species Interaction Example

Today we will investigate the interaction of two species that both have logistic growth. Each of the species is negatively effected by the interaction. We could imagine this as two species of wild cat, each of which hunts the other. Here are the equations that model this system:

$$x' = 14\left(1 - \frac{x}{7}\right)x - xy$$
$$y' = 16\left(1 - \frac{y}{8}\right)y - cxy$$

Here the numbers 14 and 16 represent the maximum reproductive rate of each species. This means that species y has a slightly higher reproduction rate than species x. The numbers 7 and 8 represent the ecological carrying capacity, meaning that the ecosystem can sustain 7 of species x and 8 of species y before the population exceeds the limited resources. The interaction term is the term we will investigate! Here the parameter c represents what happens when x and y interact. We can see that each interaction decreases x but the effect on species y depends on the value of c. We will do a fixed point analysis of this system. Here we will assume that $c > 0$

1. First find the fixed points of the system. Discuss what they mean. What are the possibilities for the value of c? In other words, when do we see a realistic fourth critical point?
2. Now analyze the fixed points using the Jacobian and Eigenvalues:
   - The first three should be straightforward. For the fourth, more complicated, fixed point use SageMath to help you find solutions.
   - Is the stability of the fixed point effected by the value of c?
   - If so, what conditions are required to make the point stable vs. unstable?
   - What does this mean in terms of the ecological system?
3. Plot a few Phase Planes to confirm or check your results.
4. Describe, overall, what you learned about the ecological system from this fixed point analysis.

A Damped Nonlinear Pendulum

The following system of equations models a pendulum or swing that has some damping.

$$x' = y$$
$$y' = -\sin(x) - cy$$

Here we are assuming a frequency of one and the term $c > 0$ represents the amount of damping in the system. The variables x and y represent the parametric representation of the pendulums position over time where x represents the pendulums angle away from rest and y represents the pendulums angular velocity. In other words, if $x = 0$ the the pendulum is hanging straight down, if $x = \pi$ the pendulum is sticking straight up, if $y = 0$ then the pendulum is at rest.

1. First find the fixed points of the system (there will be lots!). Discuss what they mean. Do the fixed points depend at all on c? Does this make sense?
2. Now analyze the fixed points using the Jacobian and Eigenvalues:
   - Here you will be able to classify your fixed points into two distinct cases: even and odd.
   - Is the stability or type of the fixed point effected by the value of c?
   - What do these results mean in terms of the pendulum system being modeled?
3. Plot a few Phase Planes to confirm or check your results. You could try using $c = \frac{1}{4}$ to get nice looking plots.
4. Describe, overall, what you learned about the pendulum system.

SPECIES INTERACTION

1. Fixed Points. set $x' = y' = 0$

$$14\left(1 - \frac{x}{7}\right)x - xy = 0 \quad \rightarrow \quad x(14 - 2x - y) = 0$$
$$16\left(1 - \frac{y}{8}\right)y - cxy = 0 \quad \quad y(16 - 2y - cx) = 0$$

	$x=0$	$14-2x-y=0$
$y=0$	$(0,0)$	$(7,0)$
$16-2y-cx=0$	$(0,8)$	$\left(\frac{12}{4-c}, \frac{32-14c}{4-c}\right)$

SOLVE FOR COEXIST POINT...

$-2\,(14-2x-y=0)$

$16-2y-cx=0$

$-12 + (4-c)x = 0 \qquad x = \frac{12}{4-c}$

then $y = 14 - 2x$

$= 14 - \frac{24}{4-c} = \frac{14(4-c)-24}{4-c}$

$= \frac{32-14c}{4-c}$

$(0,0)$ NO SPECIES

$(7,0)$ ONLY SPECIES X

$(0,8)$ ONLY SPECIES Y

coexistence point, both species can exist.

but this depends on c! $c \neq 4$ and $c > 4$

means $x < 0$ so this doesn't work.

2. Jacobian analysis. $J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 14-4x-y & -x \\ -cy & 16-4y-cx \end{bmatrix}$

Plug in FP AND analyze.

$J\big|_{(0,0)} = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \qquad |A - \lambda I| = 0 \Rightarrow \lambda = 14, 16$

NO DEPENDENCE ON c TWO POS EVALS $\lambda_1 \neq \lambda_2$ unstable improper Node.

origin always unstable.

$J\big|_{(7,0)} = \begin{bmatrix} -14 & -7 \\ 0 & 16-7c \end{bmatrix} \qquad |A - \lambda I| = 0 \Rightarrow \lambda = -14,\ 16-7c$

STABLE ONLY IF $16 - 7c < 0$

$16 < 7c$

$c > 16/7 \sim 2.286$

otherwise SADDLE. UNSTABLE.

$J\big|_{(0,8)} = \begin{bmatrix} 6 & 0 \\ -8c & -16 \end{bmatrix} \qquad \lambda = 6, -16$

SADDLE POINT Regardless of c! UNSTABLE.

The Last one is complicated! I would use SAGE!

$$J\Big|_{\left(\frac{12}{4-c},\ \frac{32-14c}{4-c}\right)}$$

ON Sage we see when $c \lesssim 2.25$ stable improper node

but when $c \gtrsim 2.25$ saddle unstable.

The stability changes as c changes! c represents an interaction term if the interaction too strongly impacts y ($c > 2.25$) then the coexistence point is no longer possible.

SAGE (CoCalc) worksheet →

Jacobian

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```
c=var('c')

# Enter your fixed points:
#x=0
#y=0

#x=7
#y=0

#x=0
#y=8

# More complicated fixed point:
A =matrix(2,2, [2,1,c,2])
b = matrix(2,1,[14,16])
SOLN = A.inverse()*b

x=12/(4-c)
y=(32-14*c)/(4-c)

J=matrix(2,2,[14-4*x-y, -x, -c*y, 16-4*y-c*x])
print("Jacobian:")
J

M=J.eigenvalues()

print("Eigenvalues:")
lam1=M[0]
lam2=M[1]

print(lam1)
print(lam2)


print("Plot Lambda vs c:")
```

```
f1(c)=M[0]
f2(c)=M[1]
MIN=-1
MAX=3
p1=plot(f1(c),(c,MIN,MAX),rgbcolor=hue(0.3),legend_label='Lambda1')
p2=plot(f2(c),(c,MIN,MAX),rgbcolor=hue(0.5),legend_label='Lambda2')
show(p1+p2)

print("")
print("Check values of c: choose c=2")
numerical_approx(lam1(c=2))
numerical_approx(lam2(c=2))

print("")
print("Check values of c: choose c=2.5")
numerical_approx(lam1(c=2.5))
numerical_approx(lam2(c=-2.5))
```

SOLN

Jacobian:

[-2*(7*c - 16)/(c - 4) + 48/(c - 4) + 14   12/(c - 4)]

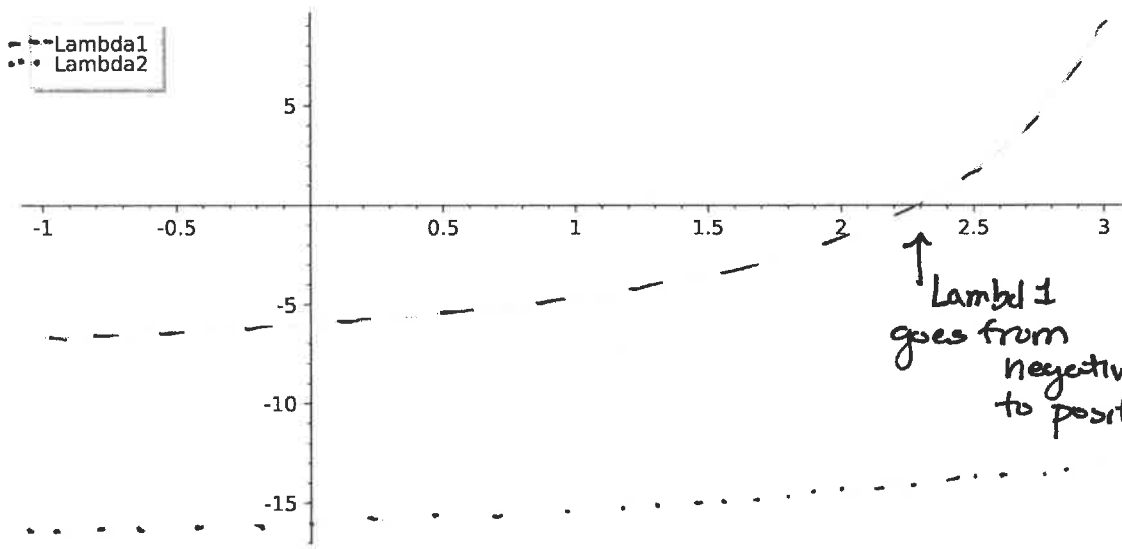
[-2*(7*c - 16)*c/(c - 4) -8*(7*c - 16)/(c - 4) + 12*c/(c - 4) + 16]

Eigenvalues:

-2*(7*c + sqrt(7*c^2 - 44*c + 100) - 22)/(c - 4)

-2*(7*c - sqrt(7*c^2 - 44*c + 100) - 22)/(c - 4) both depend on C.

Plot Lambda vs c:



Lambd 1 goes from negative to positive.

Check values of c: choose c=2

-1.67544467966324

-14.3245553203368 — improper node stable

Check values of c: choose c=2.5

1.74596669241483

-17.0552407144843

saddle point

unstable.

Check out phaze planes (Pplane) $\longrightarrow$

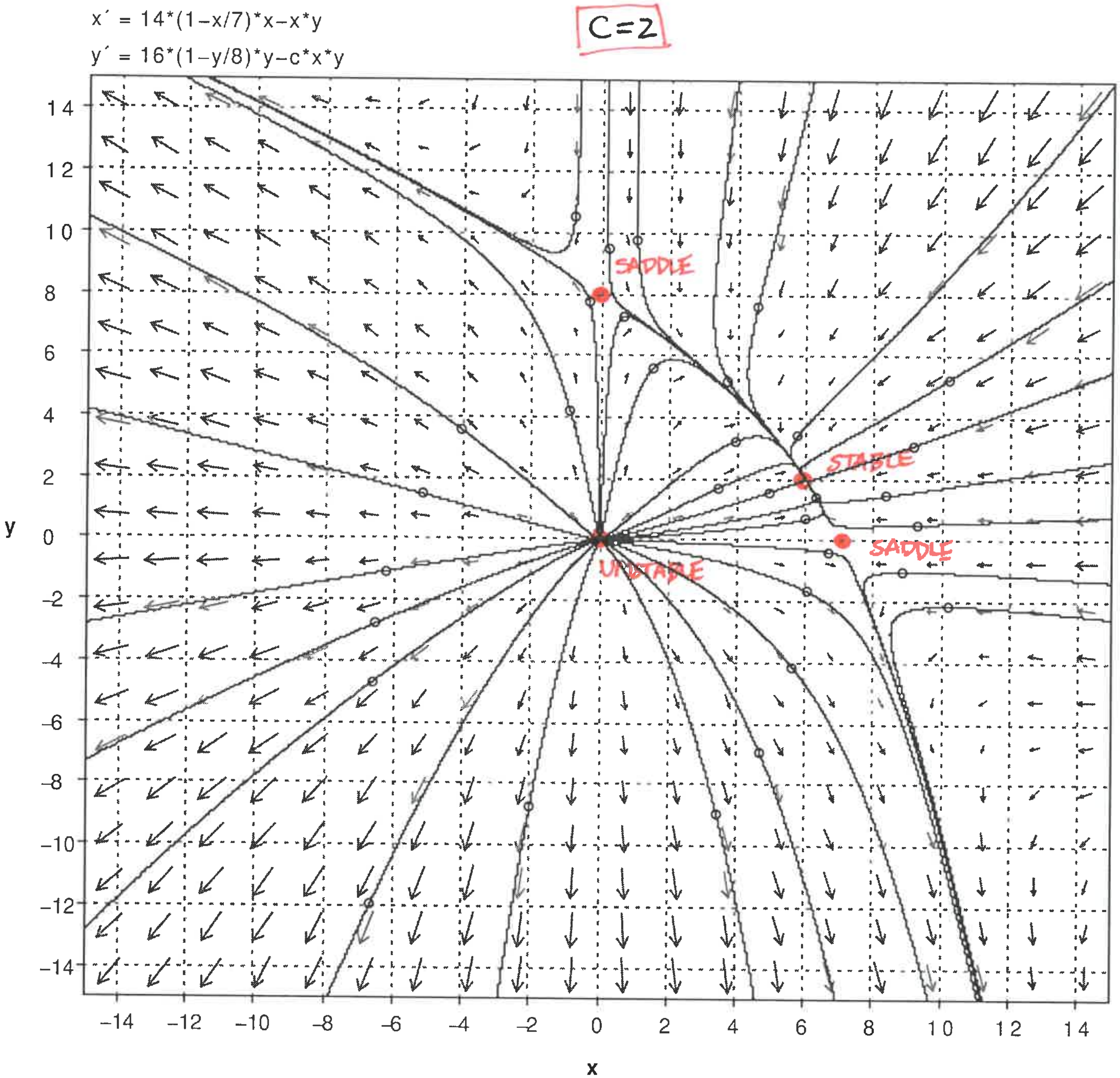
Overall. The stability of the fixed points depends on the value of c!

Coexistence is only possible when $c < 2.886$

when $c > 2.286$ the the $(7,0)$ point becomes stable and coexistence is unstable.

so when predation effects species y too much they can no longer sustain a population and we are left with only species x.

$$\boxed{C=2}$$



matches my predictions.

$c < 2.286$

$(7,0)$ is a saddle

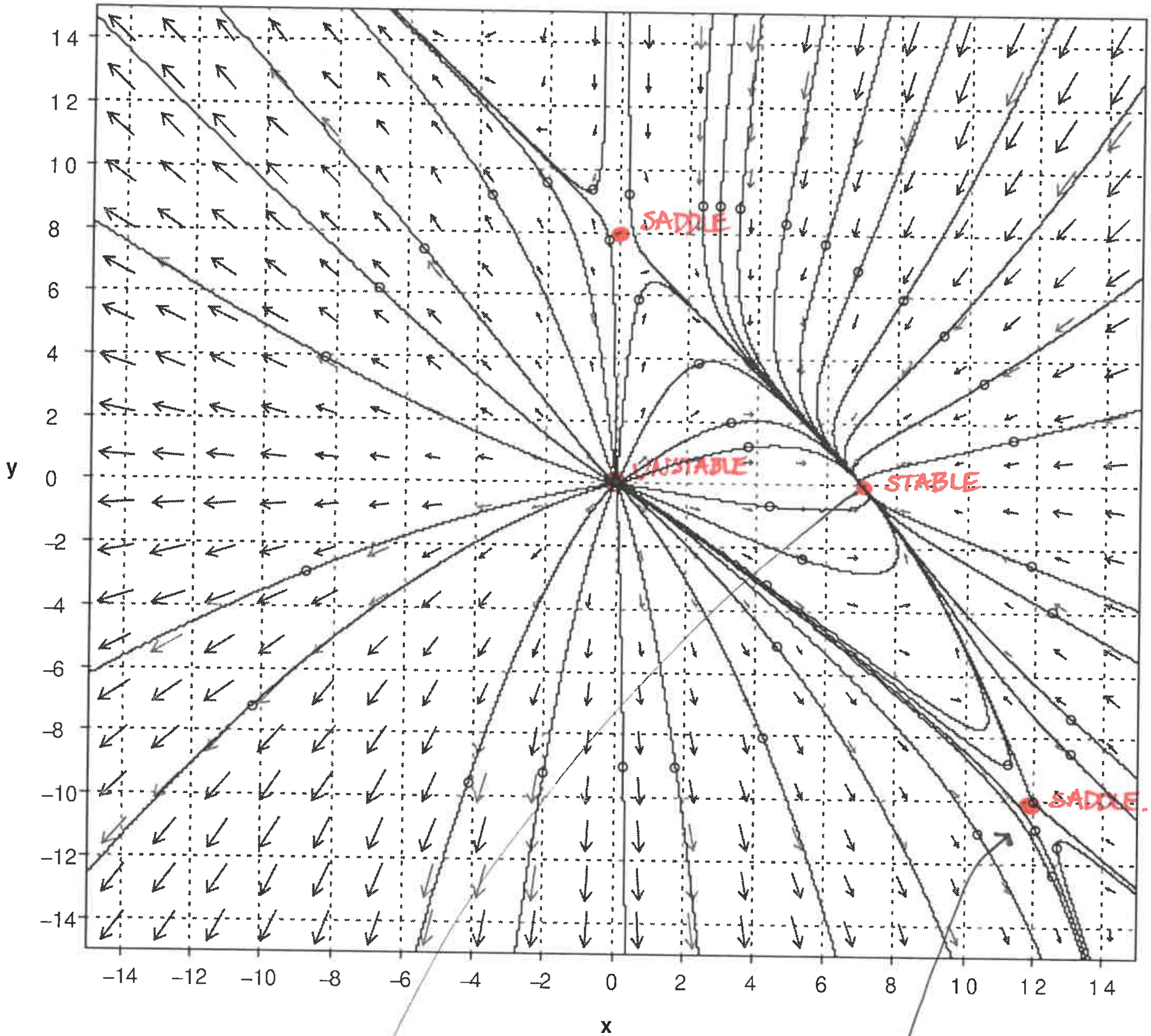
(coexist) is stable

try for $c = 3$

$\boxed{C=3}$

$x' = 14*(1-x/7)*x-x*y$

$y' = 16*(1-y/8)*y-c*x*y$



(7,0) becomes stable.

The coexistence point moves as we change c and since 3 > 2.286 it becomes unstable.

This is a Bifurcation :)

DAMPED NONLINEAR PENDULUM.

$x' = y$
$y' = -\sin(x) - cy$

$x(t)$ = angle away from rest
$y(t)$ = angular velocity.
c = constant > 0

1. Fixed Points:

$y = 0$
$-\sin(x) - cy = 0$

$y=0$
$\sin(x) = 0$ when $x = \pm n\pi$

Infinite # of fixed points: $(\pm n\pi, 0)$ $n = 0, 1, 2, 3, \ldots$

2. Jacobian $J = \begin{bmatrix} 0 & 1 \\ -\cos(x) & -c \end{bmatrix}$

$n=0 \to (0,0)$ $\quad J\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix}$ $\quad \begin{vmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{vmatrix} = -\lambda(-c-\lambda) + 1 = 0$

$= \lambda^2 + c\lambda + 1 = 0$

$\lambda = \frac{-c \pm \sqrt{c^2-4}}{2} = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2-4}$

No mater what STABLE!

If $c^2 - 4 < 0$ $\quad c \leq 2$ then decaying spirals.
because $-\frac{c}{2} < 0$ since $c > 0$

$c^2 - 4 > 0$ $\quad c > 2$ then $-\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2-4} > 0$?

This is always negative for both.

$-c \pm \sqrt{c^2-4} > 0$
$\pm\sqrt{c^2-4} > c$
$c^2 - 4 > c^2$ NO

$n=1 \to (\pi, 0)$ $\quad J\Big|_{\pi,0} = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix} = \lambda^2 + c\lambda - 1 = 0$

No mater what UNSTABLE.

$\lambda = \frac{-c \pm \sqrt{c^2+4}}{-2} = \frac{c}{2} \pm \frac{1}{2}\sqrt{c^2+4}$

Always unstable AT LEAST ONE λ is greater than zero!

$n=2 \quad J\Big|_{(2\pi,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix}$ matches $n=0$ $\longrightarrow$ For all $n=$ even we get STABLE

$n=3 \quad J\Big|_{(3\pi,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -c \end{bmatrix}$ matches $n=1$ $\longrightarrow$ For all $n=$ odd we get UNSTABLE.

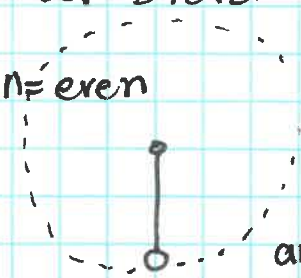
The stability is not changed by the value of c! But, for the stable nodes, $n=$ even, when $c<2$ or small damping we get decaying spirals and when $c>2$ or large damping we get improper nodes – NO OSCILLATIONS.

TRY pplane with $c=1/4$ and with $c=9/4$ $\longrightarrow$

WHAT WE LEARN:

FOR OUR SYSTEM:

n = even

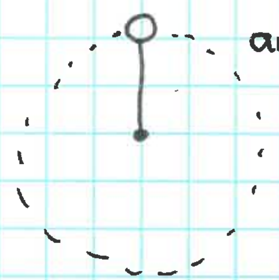


angle $= 0, 2\pi, 4\pi$

pendulum at Bottom

STABLE

n=odd



angle $= \pi, 3\pi, 5\pi$

pendulum at top

UNSTABLE

$c<2$

allows the pendulum to swing a few times.

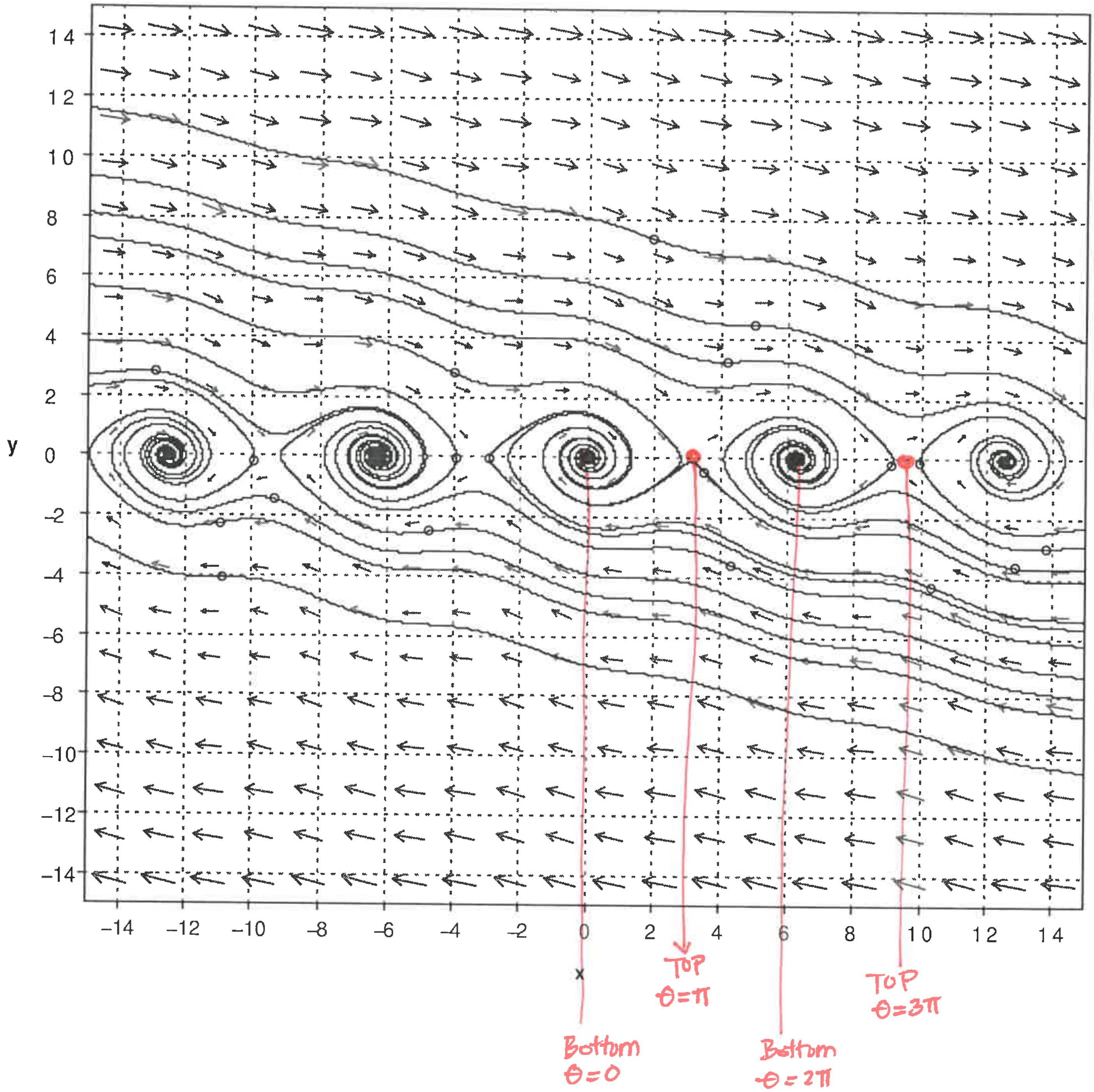
$c>2$

damping is too strong an pendulum SLOWLY Returns to FP.

$x' = y$

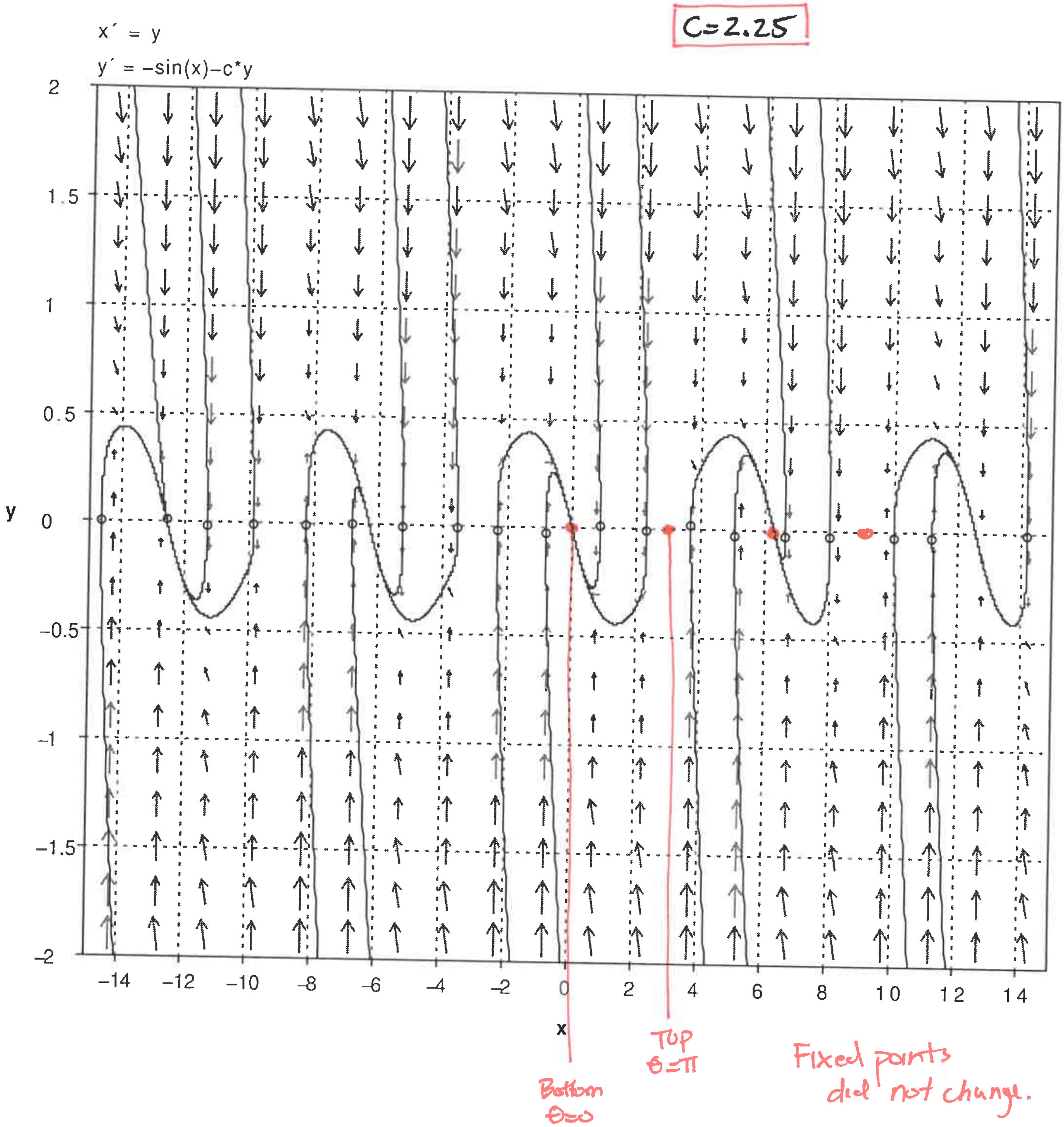
$y' = -\sin(x) - c*y$

$\boxed{C = .25}$



The decaying spirals represent oscillations of the pendulum swinging back and forth but eventually coming to rest.

$\boxed{C=2.25}$



Even nodes are still the stable ones but now there are no oscillations — pendulum just swings back to the stable F.P.