Differential Equations

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 1

What is a Differential Equation

A differential equation is really any equation that contains a derivative. It relates a rate of change to some other quantities. Basically differential equations are concerned with any type of changing quantity.

YOU TRY: Think for a second.... What things in our lives are changing quantities? Write down a function for something that changes in your life. *Eg. Joanna likes to grow a garden and her kale plants are amazing! We could write a function that tracks the growth of kale over time: K=f(t) and the amount they change over time* $\frac{dK}{dt}$.

Often things change with respect to time as we march into the future. So I like to sum up the goal of differential equations as "Predicting the future of the Universe!"

Some examples of differential equations:

$$\frac{dx}{dt} + x = 0 \tag{1}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{2}$$

$$\frac{d^2y}{dt^2} = \frac{g}{l}\sin(y) \tag{3}$$

$$\frac{d^8p}{dx^8} = 3 \tag{4}$$

You could write down your own differential equation just relate one or more derivatives and other quantities. It's that easy! Right... now how to we solve these beasts?

Classifying Differential Equations

In order to solve a differential equations you need to know what kind of beast you are dealing with. You don't want to bring a llama to a snake fight! Especially when you are just starting to learn differential equations, clearly classifying your equation is one of the most important parts and helps determine a path toward the solution.

1. **The Order:** This is just the degree of the highest derivative.

   Recall that this is the "power" of the derivative so $\frac{dy}{dt}$ is first order while $\frac{d^3y}{dt^3}$ is third order

   Looking back at our example, can you figure out the order of each equation? [1]

2. **Linear vs. Nonlinear:** Oh the importance of being linear! Similar to algebra, we look to see if dependent variable is raised to a power or inside a function.

   Recall, if we have a function $y = f(x)$ then the value of y depends on what we "plug in" for x. So x is the independent variable and y is the dependent variable. Now, what if we took a good old fashioned derivative? We would write $\frac{dy}{dy}$ so you can tell in the derivative which is dependent and which is independent.

   Some Linear Examples:

   $$\frac{d^2p}{dt^2} = 7t^2$$
   $$\frac{ds}{dx} + e^s = 0$$

   Can you say which is the dependent and which is the independent variable? Don't those nasty functions make these things nonlinear?

   Some Nonlinear Examples:

   $$\frac{d^2p}{dt^2} = 7p^2$$
   $$s\frac{ds}{dx} + e^s = 0$$

   But wait! These look almost exactly like the linear examples. What has changed?

[1] Eq. 1 is first order, Eq. 2 is second order, Eq. 3 is second order, and Eq. 4 is eighth order.

YOU TRY: Write down 3 linear and 3 nonlinear differential equations

Looking back at our example, can you say which differential equations are linear or nonlinear? [2]

3. MORE EXAMPLES

   Classify the following differential equations

$$\left(\frac{d^2y}{dt^2}\right)^2 + y = 0$$

   This would be second order, nonlinear, with an independent variable t and a dependent variable y.

$$\frac{ds}{dp} + s + \sin(p^3) = 0$$

   This would be first order, linear, with an independent variable p and a dependent variable s.

4. **Ordinary vs. Partial:**In this class we study Ordinary Differential Equations (ODEs) which are equations with functions of one variable $y = f(t)$. Remember from calculus when you learned about partial derivatives? That was for functions of two variables $y = f(x,t)$. Same thing goes for Partial Differential Equations (PDEs). Basically if you have a function of more than one variable or you see a partial derivative you are dealing with a partial differential equation.

   A quick example of a PDE:

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}$$

   This is a simple Heat Equation. Here you see partial derivatives and the variable θ, representing heat, depends on both space, x, and time t.

`What do solutions look like?`

Well great, we can write down and classify these things, but what are their solutions? Hopefully we answer that questions in excruciating detail during this semester! Here is a simple example:

Let's pick a nice function out of our sack of function... how about $y = ce^{x^2}$ where c is some constant. Now what if we took the derivative of this thing with respect to x. [3]

$$\frac{dy}{dx} = 2x\left(ce^{x^2}\right)$$

But wait! That thing in parenthesis is just our original function y! So we could write

$$\frac{dy}{dx} = 2xy \tag{5}$$

That means that Eq. 5 is a first order linear ODE and it's solution is $y = ce^{x^2}$. In the real world we would still need to determine that unknown constant, meaning we need an additional piece of information, usually an initial condition.

Initial conditions are the additional pieces of information that help you identify the constants of integration that result from solving a differential equation. Why do these constants happen? Well, let's think of a real world example:

We start with a differential equation or rate of change

Joanna drives in a straight line going 100 miles per hour on her way to work! $\frac{dx}{dt} = 100$ where x is Joanna's position and t is time

Then we ask a question:

If Joanna leaves at 8am, where with she be at 8:15?

This is impossible to answer until we have an additional piece of information, right? You have to know where I started, or how far away from school I live. This is the initial condition:

Joanna lives 5 miles away from campus so at time zero $t = 1$ she is -5 from campus, or $x(0) = -5$

This meas that by 8:15 I am either already to campus or arrested for reeking havoc on the streets of Redlands.

[2] Eq. 1 is linear because there are no powers or functions, Eq. 2 is linear higher derivatives and multiplying by constants doesn't effect linearity, Eq. 3 is nonlinear because the dependent variable y is inside the function $\sin(y)$, and Eq. 4 is linear.

[3] Here is the derivative in extreme detail as a reminder of the chain rule:

$$\frac{dy}{dx} = c\left(\frac{d}{dx}e^{x^2}\right) = ce^{x^2}\frac{d}{dx}\left(x^2\right) = ce^{x^2}2x = 2x\left(ce^{x^2}\right)$$

Differential Equations have Physical Meaning!

We could spend a whole semester just listing all the applications of differential equations! But here are some to give you and idea.

1. Newtons Law of Cooling - AKA How to mathematically bake a turkey:

$$\frac{dT}{dt} = -k(T - A)$$

   where A is the ambient temperature (the oven) and k is the thermal constant assumed positive (how easily the turkey gains or loses heat). So if the turkey is colder than the oven, will the temperature of the turkey, T, increase or decrease? Increase both logically, having interacted with an oven before, and from looking at the equation!

2. Torricelli's Law - AKA How to mathematically shotgun a soda:

$$\frac{dV}{dt} = -k\sqrt{y}$$

   where k is the flow constant and here we would relate the volume V of the container and how the depth of fluid y changes depending on the shape of the container. Here we see that as the depth of fluid decreases so does the rate at which the volume decreases. Less soda in that can means the amount streaming out decreases.

3. Population Dynamics - AKA How much bacteria is really on that pizza slice that I left out over night but still ate for breakfast this morning:

$$\frac{dP}{dt} = kP$$

   where k is the growth rate and P is the number of members of the population. As more bacterial colonizes my pizza, there are more members of the population to reproduce, and thus the overall population growth rate is more, assuming an infinite pizza and a few other simplifying things.

Review of Integration

Every time you take another class in the math sequence I sincerely hope you take the opportunity to relearn the material from past classes. It really is okay if you don't remember immediately how to do some piece of algebra, trigonometry, or calculus, but now is the time to relearn it! For me, I often have to bump into something 3-4 times in different settings before I feel like I really get it.

Here is a quick and dirty review of integration, or how I think of doing integration. I hope it's helpful.

The Five Basic Methods

1. Just know the darn solution!
2. Substitution
3. Integration by Parts
4. Algebra: Partial Fractions, Completing the Square, Polynomial Long Division
5. Trigonometric Substitution and Trigonometric Tricks

I usually think of integration in this order. I always hope that I just know the darn solution or that I can easily simplify the answer to one I just know. It is worth a few seconds of your time to see if you can simplify the equation before jumping into one of the more involved methods!
Example: Just do it!

$$\int \frac{x^2 \sin(x)}{\sin(x)}\, dx$$

Okay, my first instinct: *Run for the hills there is a $sin(x)$ in the denominator!!!!* But then sanity kicks in and I realize that I can do some simple algebra here.

$$\int \frac{x^2 \sin(x)}{\sin(x)}\, dx = \int x^2 dx = \frac{1}{3}x^3 + c$$

Example: Substitution
I always choose substitution as a possible method when I see a function inside a function in my integrand, for integrals of the form

$$\int f(g(x)) \cdot g'(x) dx$$

An example of this is

$$\int (2x+1)e^{x^2+x} dx$$

First, I notice e^{x^2+x}, this is an example of a function inside a function. Then I look at the integrand (stuff between the integral sign and the dx and ask "What is the most annoying part of this integral?" Well, I could easily integrate $(2x+1)$ but the annoying part is e^{x^2+x} so I will try to make that part easier by substituting something for the inner function! This helps me choose the substitution:

$$u = x^2 + x$$
$$du = (2x+1)dx$$

and for some it helps to solve for $dx = \dfrac{du}{(2x+1)}$. Now substitute all these parts in

$$\int (2x+1)e^{x^2+x}dx = \int (2x+1)e^u \frac{du}{(2x+1)}\ dx$$
$$= \int e^u du$$

Some people skip the middle step and just see $(2x+1)dx = du$ in the integral. Well, now I've transformed this into one I just know, yay!

$$\int (2x+1)e^{x^2+x}dx = \int e^u du = e^u + c = e^{x^2+1} + c$$

Some things to check

- After substitution you should have no more of your old dependent variable left. In this case all of the x's are gone and replaced by u's
- Don't forget the last step of getting back to your original variables!

Example: Integration by Parts
This method is great when you see two functions multiplied that can be simplified and do not contain a function inside a function, for integrals of the form

$$\int f(x) \cdot g(x)dx$$

An example of this is

$$\int w \sin(w)\ dw$$

Here is see a w and a $\sin(w)$ independently, neither of these functions would be terrible to integrate, it is the fact that they are multiplied that makes them troublesome. So I will use integration by parts. Here is our trusty formula:

$$\int u\ dv = uv - \int v\ du$$

So we need to choose a u and a dv then take the derivative of u to get du and the integral of dv to get v. Here goes, I will choose

$$u = w \qquad dv = \sin(w)dw$$

Now you ask "Why did you choose it that way?" and I say "Because I am a math genius and only I can predict the u's and the dv's, pay me lots of money!" NO! Here are the rules:

- Pick something for u that gets simpler when you take the derivative. For example, the derivative of x is 1 a good choice for u. Usually functions like x, x^2, x^n and $\ln(x)$ make good choices for u
- Pick something for dv that doesn't get harder when you integrate. For example, the integral of $\sin(x)$ is $-\cos(x)$ as good choice for dv. Usually functions like $\sin(x)$, $\cos(x)$, and e^x are good choices for dv.

Okay moving on, now we need the other pieces of our formula:

$$u = w \qquad dv = \sin(w)dw$$
$$du = 1 \cdot dw \quad v = -\cos(x)$$

Now apply the formula (AKA plug that stuff in!)

$$\int w \sin(w)\ dw = -w\cos(w) - \int (-\cos(w))\ dw$$
$$= -w\cos(w) + \sin(w) + c$$

Notice that after doing integration by parts the integral part got easier, in fact it was one I just know, yay!

Some things to check:

- Integration by parts is often over used! So make sure that after you use the formula you actually get a simpler integral. All these integrations methods should make your life easier!
- Sometimes the integral gets easier, aka reduces the power of the polynomial, but the polynomial is still there. This means we might need to do integration by parts more than one time. Ugggg! Noway, think of it like meditation... Ommmmm Innnntaaagraaation by Paaaaarts! Here is an example:

  $$\int x^2 \cos(x)\ dx$$

  we would do integration by parts twice. How many times would we integrate by parts for

  $$\int x^{16}e^x\ dx$$

  You guessed it, a lovely and relaxing 16 times!

Example: Algebra Partial Fractions
It is worth your time to look online (Kahn Academy, Wolfram, etc) for a quick tutorial in basic algebra. You want to be the boss of these techniques to make your life as easy as possible. Here is a list of good techniques to master early this semester

- Factoring and Multiplying Polynomials - FOIL and friends
- Polynomial Long Division
- Partial Fraction Decomposition
- Completing the Square
- Solving systems of linear equations

Here is an example using one of these methods to show how nice it is to boss around algebra.

$$\int \frac{5}{(2x+1)(x-2)}$$

Here I can't easily simplify, I don't see a function inside a function, and there are not two nicely multiplied functions so my above techniques wont work so well. What I do see is a big ole' polynomial in the denominator... time to dust off high school algebra! This integral would be way easier if I could just break up those denominators. My dream is to be able to write

$$\frac{5}{(2x+1)(x-2)} = \frac{A}{(2x+1)} + \frac{B}{(x-2)}$$

Well since I am the boss lets solve this equation! [4]
Multiply through by the LHS (left hand side) denominator

$$5 = A(x-2) + B(2x+1)$$

simplify the RHS (right hand side) by gathering like powers of x

$$5 = (A+2B)x + (B-2A)$$

the force the equation to balance! I have zero x's on the LHS so $0 = (A+2B)$ and I have 5 ones on the LHS so $5 = (B-2A)$. This gives me two equations for two unknowns.

$$0 = (A+2B) \rightarrow A = -2B$$
$$\text{plugging into the other equation}$$
$$5 = B - 2(-2B) = B + 4B = 5B \quad \text{so} \quad B = 1$$
$$\text{plugging into the first equation}$$
$$A = -2(1) = -2$$

This means I can rewrite my original integral

$$\int \frac{5}{(2x+1)(x-2)} = \int \frac{-2}{(2x+1)}\,dx + \int \frac{1}{(x-2)}\,dx$$

These can be solved either by substitution or by being really good at integrating $1/x$ in your head.

$$\int \frac{-2}{(2x+1)}\,dx + \int \frac{1}{(x-2)}\,dx = -\ln|2x+1| + \ln|x-2| + c$$

Example: Trigonometric Substitutions
I usually know to use a trigonometric substitution (Trig Sub) when I see terms like $\sqrt{a^2 \pm b^2x^2}$ or $\frac{1}{(a^2+b^2x^2)^n}$. These things remind be of parts of Pythagorean Theorem $a^2 + b^2 = c^2$ so we use triangles to find a substitution.

Here is an example

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

Here I see in the denominator the Pythagorean theorm if we solved for $a^2 = c^2 - b^2$ so I can construct a triangle almost always putting my variable x across from θ.

IMAGE HERE

And using SOH CAH TOH I read the substitution from the triangle. Here are the things I read:

1. The most simple thing is my substitution
$$x = \sin(\theta)$$
2. Take the derivative of the substitution to get
$$dx = \cos(\theta)\,d\theta$$
3. The read the $\cos(\theta)$ to get something to simplify the integrand
$$\cos(\theta) = \sqrt{1-x^2}$$
4. Now we substitute all of this into the integral

[4] Often the hardest part of partial fractions is deciding how to separate the denominator. There are special rules for repeated roots, for example $\frac{1}{(x+2)^2(x-1)}$ we CANNOT break this into $\frac{A}{(x+2)} + \frac{B}{(x+2)} + \frac{C}{(x-1)}$ since the first two fractions would just add together! So more generally the format is

$$\frac{1}{(x-1)(x+2)^2(x^2+x+1)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{Dx+E}{x^2+x+1}$$

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{(\sin(\theta))^3}{\cos(\theta)} \cdot \cos(\theta)\,d\theta$$

At this point things should simplify!

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \sin^3(\theta)\,d\theta$$

This is actually a much easier integral if you know the rules for integrating powers of sine and cosine!

Powers of Sine and Cosine
There are two basic cases for powers of sine and cosine.

1. **EVEN POWERS** Separate the functions into powers of two and then use the double angle formulas.

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$$
$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

2. **ODD POWERS** Separate out the one odd power and then rest into powers of two. Then use

$$\sin^2(x) + \cos^2(x) = 1$$

   This should set you up for a nice substitution!

Back to our previous example

$$\int \sin^3(\theta)\,d\theta = \int \sin(\theta)\cdot\sin^2(\theta)\,d\theta$$

now from our odd powers rule

$$\sin^2(\theta) = 1 - \cos^2(\theta)$$

and

$$\int \sin(\theta)\cdot\sin^2(\theta)\,d\theta \int \sin(\theta)\cdot(1-\cos^2(\theta))\,d\theta$$

We can then substitute $u = \cos(\theta)$

$$\int \sin(\theta)\cdot(1-\cos^2(\theta))\,d\theta = \int 1-u^2\,du = u - \frac{1}{3}u^3 + c$$

So our answer would be

$$\int \sin^3(\theta)\,d\theta = \cos(\theta) - \frac{1}{3}\cos^3(\theta) + c$$

And putting all the pieces together from the very beginning of our trigonometric substitution example... what were we doing again... oh right... solving

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

so we need to get our equation back in terms of x! Reading directly from the triangle, or looking at the things we used in the substitution, we remember that

$$\cos(\theta) = \sqrt{1-x^2}$$

so just plug that in everywhere that you see a $\cos(\theta)$

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$$

The Moral(s) of the Story

- Differential equations are awesome and they have tons of real world applications!
- Your first job for every ODE, for the rest of the semester, is to classify it. Just do it! Yes, every time!
- Integration can be tricky and that's why it's fun. Remember that your goal with all these methods is to make your life EASIER not harder. Our goal is always to turn a nasty looking integral into one we just know and love.
- It's never too late to relearn methods or topics from algebra, trigonometry, or calculus. The only way to get yourself into trouble is to give up and quit trying. Give the you of the future a HUGE gift that is learning the foundations NOW.