Differential Equations - Notes

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 11
The goal today is to define and understand some important terms that we will be using for the rest of the class. Please make sure that you understand these ideas and come get help if you need it!

Four BIG Ideas for Linear Equations!

We will be focusing on linear equations for the next few weeks and for our discussion today we will consider second order linear equations.
YOU TRY:
Which of these is a second order linear ODE?

$$e^x y'' + \cos(x) y'_(1 + \sqrt{x}) y = \arctan(x)$$

$$y'' + 3(y')^2 + 4y = 0$$

ANSWER [1]
You should see that the first equation is linear while the second equation is nonlinear because it contains the dependent variable in a function, namely $(y')^2$

Here are four ideas that we will use for the rest of the class:

1. Homogeneous vs. Nonhomogeneous
2. The Principle of Superposition
3. The Wronskian Test for Linear Independence
4. The General Solution

We will focus, for now, on second order linear equations of the form:

$$A(x)y'' + B(x)y' + C(x)y = F(x), \quad y(a) = b_0 \; y'(a) = b_1$$

another way to write the ODE would be

$$y'' + p(x)y' + q(x)y = r(x)$$

As long as A, B, C, and F are continuous on some open interval containing the point a then solutions exist and are unique.

[1] The first one is! The second one has nonlinear term $(y')^2$

ASIDE *Before we continue lets think about an important question... Why do we care about higher order equations?*

Well, simply, in the real world we care about acceleration!

EXAMPLE:
The Mass Spring Equation:
In a frictionless mass spring system moving only in the x-direction with a shock absorber we have the following ODE:

$$mx'' + cx' + kx = 0$$

Here the force of the spring is proportional to the displacement $F_s = -kx$, the force the the shock absorber is proportional to velocity $F_r = -cv$ and we use Newtons Law $F = ma$ to get $F_s + F_r = ma$. If we realize that $a = x''$ and $v = x'$ we are left with a second order linear ODE.

One interesting fact... We cannot get oscillating solutions using a single first order ODE!

Okay now on the the important definitions!

1. **HOMOGENEOUS vs. NONHOMOGENEOUS**
   In our general form if $F(x) = 0$ then the ODE is Homogeneous and if $F(x) \neq 0$ then the ODE is Nonhomogeneous

   Homogeneous equations are "unforced" in other words the acceleration, velocity, and position relationship tell us all we need to know. In a Nonhomogeneous equation we have a "forcing term" on the RHS. This means that there is some external push in the system. In the spring mass system this is like giving the spring a little extra push during the oscillations.

YOU TRY: Classify the following systems if linear say whether they are Homogeneous or Nonhomogeneous:

$$y'' + x^3 y + 3y = 0$$
$$yy'' + y - 2y^2 = 0$$
$$y'' + \sqrt{x}y' + y = x^2 + 3$$
$$y'' + 3y' + y - 4 = 0$$

If the equation is Nonhomogeneous, then what is the Nonhomogeneous term? ANSWERS [2]

2. **THE PRINCIPLE OF SUPERPOSITION**

Theorem 1 *Let y_1 and y_2 be solutions to a linear homogeneous ordinary differential equation on the interval I. If c_1 and c_2 are constants then the linear combination*

$$y = c_1 y_1 + c_2 y_2$$

is also a solution to the ODE.

Let's prove this theorem. First we assume that y_1 and y_2 are solutions to a linear homogeneous ODE. In other words

$$y_1'' + p(x)y_1' + q(x)y_1 = 0$$

and

$$y_2'' + p(x)y_2' + q(x)y_2 = 0$$

Then consider the solution $y = c_1 y_1 + c_2 y_2$. If this is a solution it must satisfy the ODE, meaning that we can plug into the ode:

$$[c_1y_1 + c_2y_2]'' + p(x)[c_1y_1 + c_2y_2]' + q(x)[c_1y_1 + c_2y_2$$

but rearranging terms and taking derivatives gives

$$\begin{aligned} &c_1y_1'' + c_2y_2'' + p(x)c_1y_1' + p(x)c_2y_2' + q(x)c_1y_1 + q(x)c_2y_2 \\ &\quad c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2'' + q(x)y) \end{aligned}$$

But because we already assumed that y_1 and y_2 were solutions we find

$$c_1(0) + c_2(0) = 0$$

Or, that the ODE is in fact satisfied by $y = c_1y_1 + c_2y_2$, so y is also a solution.

3. **THE GENERAL SOLUTION**
We write our general solution to a linear ODE as the sum of all possible solutions. If y_1 and y_2 are both solutions then our general solution is

$$y = c_1 y_1 + c_1 y_2$$

We then use initial conditions to find c_1 and c_2 giving us our particular solution.

EXAMPLE:
Consider the equation

$$y'' + y = 0$$

This equation has the solutions $y_1 = \sin(x)$ and $y_2 = \cos(x)$ we would write the general solution as

$$y = c_2 \sin(x) + c_2 \cos(x)$$

here the order doesn't mater. Then assume that we have $y(0) = 1$ and $y'(0) - 1$, we can find a particular solution.

$$\begin{aligned} c_1 \sin(0) + c_2 \cos(0) = c_2 = 1 \\ c_1 \cos(0) - c_2 \sin(0) = c_2 = 1 \end{aligned} \tag{1}$$

so our particular solution is $y = \sin(x) + \cos(x)$

Something sneaky happened behind the scenes here!

$$\begin{bmatrix} sin(0) & cos(0) \\ cos(0) & sin(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In order to solve for c_1 and c_2 you the inverse must exist or the determinate of the 2×2 matrix above cannot be equal to 0. This also means that our two solutions much be linearly independent!

4. **LINEAR INDEPENDENCE and the WRONSKIAN**

In order to use the Principle of Superposition and write down a general solution to our ODE, our two solutions must be linearly independent. What does it mean to be linearly independent? Simply, if the functions g and f are linearly independent the we can write one as a constant multiple of the other.

EXAMPLE:
Linearly independent functions like $\sin(x)$ and $\cos(x)$ cannot be written in terms of the other one times a constant. Another way to see this is $\frac{\sin(x)}{\cos(x)}$ is not a constant. So, if they are solutions to our second order linear homogeneous equation, then our general solution is written as $y = c_2 \sin(x) + c_2 \cos(x)$.

EXAMPLE:
Linearly dependent functions like x and $8x$ can be written in terms of the other one times a constant. We either multiply x by 8 or we divide $8x$ by 8. Another way to see this is $\frac{8x}{x} = 8$ and this is a constant! If we claimed that these were both solutions to our second order linear homogeneous equation, then our general solution would be written as $y = c_1x + c_2 8x = (c_1 + 8c_2)x = Ax$ so really we only ever found a single solution!

[2] 1. Homogeneous, 2. Nonlinear, 3. Nonhomogeneous $F(x) = x^2 + 3$, 4. Nonhomogeneous $F(x) = 4$

The Wronskian is a quick test for the linear independence of functions:

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'$$

If $W = 0$ then our functions f and g are linearly dependent. If $W \neq 0$ then our functions f and g are linearly independent.

EXAMPLE:
Consider the functions $f = e^x$ and $g = xe^x$. Use the Wronskian to check if these functions are linearly dependent.

$$\begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

So the Wronskin $W(e^x, xe^x) = e^{2x}$, this is not equal to zero so these functions are linearly independent.

YOU TRY:
Test the functions from our example for linear independence using the Wronskian:

$$y_1 = \sin(x) \quad y_2 = \cos(x)$$

$$y_1 = x \quad y_2 = 8x$$

ANSWERS [3]

The Moral(s) of the Story

- Linear second order ordinary differential equations can be homogeneous or nonhomogeneous. This is a very important distinction.
- If our linear second order ODE is homogeneous and we have two solutions y_1 and y_2 then we can write the general solution using the principle of superposition: $y = c_1y_1 + c_1y_2$
- It is important that our two solutions in the superposition are linearly independent and we can use the Wronskian test to check for linear dependence.

[3] $W(\sin(x), \cos(x)) = -1$ so these are linearly independent, $W(x, 8x) = 0$ so these are linearly dependent