Differential Equations - Notes

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 12
We are continuing our discussion of linear higher order equations.

How many solutions should we expect?

Last time we talked about how, for linear homogeneous equations, if we have multiple solutions then we write the general solution as the sum of those solutions. This means that the most general solution is the sum of ALL POSSIBLE solutions. But, how would we know that we found all possible solutions?

It turns out that an n^{th} order linear homogeneous ODE has exactly n unique linearly independent solutions. So for our second order equation we would expect exactly two solutions, once we find two we have them all. Let's see how we would prove this. This can be easily generalized for an n^{th} order equation.

Recall the form of our second order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = r(x), \quad y(a) = b_0 \;\; y'(a) = b_1$$

We begin our proof by assuming that we have found one solution.
Let y_1 and y_2 be linearly independent solutions to our second order homogeneous ODE. Then write

$$y = c_1y_1 + c_2y_2$$

as a general solution that satisfies the initial conditions $y(a) = b_0$ and $y'(a) = b_1$, where a is on some open interval I. Then we can solve for the constants using

$$\begin{array}{l} c_1y_1 + c_2y_2 = b_0 \\ c_1y_1' + c_2y_2' = b_1 \end{array} \rightarrow \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

So we find that

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

Notice that the 2×2 matrix being inverted here here is similar to the Wronskian and that we are guaranteed solutions to this equation because y_1 and y_2 are linearly independent. Now assume that we can find some other solution that satisfies the ODE, $F(x)$ and write our new general solution as

$$y = c_3F + c_1y_1 + c_2y_2$$

Let us apply the initial conditions to the new general solution

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} c_3F(a) \\ c_3F'(a) \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

From our work above we see that

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} + \begin{bmatrix} c_3F \\ c_3F' \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

So we must choose either $c_3 = 0$ or $F(x) = 0$ but either way this leaves us with a general solution of

$$y = c_1y_1 + c_2y_2$$

and with the original two linearly independent solutions we have found all solutions to the linear second order homogeneous ODE.

All of what we have done can be generalized to an n^{th} order linear ODE with general form

$$P_0y^{(n)} + P_(1)y^{(n-1)} + \cdots P_{n-1}y' + P_ny = F(x)$$

where we would seek n linearly independent solutions and then write the general solution as

$$y(x) = c_1y_1 + c_1y_1 + \cdots c_ny_n$$

We can still use the Wronskian

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

All of this is great news! We know exactly how many solutions we need to find. But how do we find these solutions? We could just stab around in the dark and hope for the best! But, as mathematicians, lets see if we can come up with a better method.

Finding Solutions to Linear Constant Coefficient ODEs

Again we will focus on second order, even though all of this can easily be done for higher order. Consider the equation

$$ay'' + by' + cy = 0$$

What kind of functions have the behavior we see in this ode? In other words what kind of functions, when you take their derivative, just kick out a constant and otherwise remain unchanged?

Have you thought about it?

Seriously, write down a few functions and see if you can guess before I tell you!

If we think about the exponential function it does just what we want.

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

So lets just assume for now that ALL linear homogeneous constant coefficient equations have solutions of the form

$$y = e^{rx}$$

This is our **ansatz**! Hey cool new math word! What does it mean? An ansatz is an assumed form for a mathematical statement that is not based on underlying theory. In other words it is a complete guess that is guided by mathematical intuition. We usually do this in differential equations to seek new solution methods or substitutions, we assume a form, plug that form into the equation, and then see what mathematics has to say about it.

Here we will plug $y = e^{rx}$ into the form for our general second order ODE

$$\begin{aligned} ar^2 e^{rx} + bre^{rx} + ce^{rx} &= 0 \\ e^{rx}\left[ar^2 + br + c\right] &= 0 \\ e^{rx} \neq 0 \qquad \left[ar^2 + br + c\right] &= 0 \end{aligned}$$

This gives us specific conditions for r using the constants a, b and c. In other words we just solve the quadratic equation $ar^2 + br + c = 0$ to find values of r. And, nicely enough, we are guaranteed exactly two solutions to this equation thanks to the Fundamental Theorem of Algebra. Note these solutions could be Real, Repeated, or Complex. We call this quadratic equation **THE CHARACTERISTIC EQUATION**

So from this equation we find r_1 and r_2 giving us, we hope, two linearly independent solutions $y_1 = e^{r_1}$ and $y_2 = e^{r_2}$ so we write our general solution as $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ for now we will assume that $r_1 \neq r_2$.

Challenge Thoughts:
1. What would happen if $r_1 = r_2 = r$? Could we still write our solution as $y = c_1 e^{rx} + c_2 e^{rx}$? What would the Wronskian have to say about these solutions?
2. How would we generalize this to an n^{th} order system?

Next we will consider how we use this idea to solve real problems. We will break our solution methods up into different cases:

1. The Characteristic Equation has Distinct Real Roots
2. The Characteristic Equation has at least one Repeated Real Root
3. The Characteristic Equation has at least two Complex or Imaginary Roots

Distinct Real Roots

For the case of distinct real roots

$$r_1 \neq r_2 \neq \cdots \neq r_n$$

we can simply write our general solution as

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots = c_n e^{r_n x}$$

EXAMPLE:

$$y'' + y - 6, \quad y(0) = 1 \;\; y'(0) = 0$$

1. Write down the characteristic equation.

   $$r^2 + r - 6 = 0$$

   NOTE: We can read this directly from the ODE, notice that the coefficient in front of y'' is always the coefficient in front of r^2 and so on. If you are unsure of this step, plug in the ansatz $y = e^{rx}$ and do the algebra, you will get to the same conclusion.

2. Solve for all possible roots.

   $$(r-2)(r+3) = 0$$

   so

   $$r_1 = 2, \quad r_2 = -3$$

   It would have been equally correct to use the quadratic formula here!

3. Write down the general solution.

   $$y(x) = c_1 e^{r_1 x} c_2 e^{r_2 x} = c_2 e^{2x} + c_2 e^{-3x}$$

4. If you are given initial conditions, apply them.

$$y(0) = c_1 + c_2 = 1$$
$$y'(0) = 2c_1 - 3c_2 = 0$$

- You could solve using basic algebra:
  From the second equation
  $$c_1 = \frac{3}{2}c_2$$
  Subbing into the first
  $$\frac{3}{2}c_2 + c_2 = 1 \to c_2 = \frac{2}{5}$$
  and finally solving for
  $$c_1 = \frac{3}{2} \cdot \frac{2}{5} = \frac{3}{5}$$
  so our solution is
  $$y(x) = \frac{3}{5}e^{2x} + \frac{2}{5}e^{-3x}$$
- You could use matrix algebra and or row reduction:
  $$y(0) = c_1 + c_2 = 1$$
  $$y'(0) = 2c_1 - 3c_2 = 0$$
  In matrix form we can write
  $$\mathbf{A}\vec{x} = \vec{b}$$
  where
  $$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
  So our solution is
  $$\vec{x} = \mathbf{A}^{-1}\vec{b}$$
  $$\vec{v} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$$
  So we find that $c_1 = \frac{3}{5}$ and $c_2 = \frac{2}{5}$, same as what we got above!

EXAMPLE:
Let's try a higher order example

$$y^{(3)} + 3y'' - 10y' = 0$$

There is the straightforward method way of solving this which I will show here, but you could also be sneaky and realize that reduction of order could make this easier as a second order ODE.

How many initial conditions would we need to uniquely specify a solution? Thats right! THREE

$$y(0) = 7, \quad y'(0) = 0, \quad y''(0) = 70$$

1. Write down the characteristic equation.
$$r^3 + 3r^2 - 10r = 0$$
2. Solve for all possible roots.
$$r(r+5)(r-2) = 0$$
giving $r_1 = 0,\ r_2 = -5,\ r_3 = 2$

It is a good idea to brush up on root finding! Remember that for a general cubic, you first guess to find one root, r_1 *then use long division to factor out* $(1 - r_1)$ *leaving you with* $(1 - r_1)$*(a quadratic).*
3. Write down the general solution.
$$y(x) = c_1 + c_2e^{-5x} + c_3e^{2x}$$
4. If you are given initial conditions, apply them.
This is where it becomes handy to how to use a matrix inverse. In this class I am okay with you using a calculator, or better yet a computer algebra system like sage or matlab, to find matrix inverses.
$$y(0) = c_1 + c_2 + c_3 = 7$$
$$y'(0) = -5c_2 + 2c_3 = 0$$
$$y''(0) = 25c_2 + 4c_3 = 70$$
or in matrix form
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 25 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 70 \end{bmatrix}$$
Either solving with a matrix inverse or by substitution or by row reduction you should find
$$c_1 = 0,\ c_2 = 2,\ c_3 = 5$$
give our final solution as
$$y(c) = 2e^{-5x} + 5e^{2x}$$

YOU TRY:

$$y'' - 4y = 0$$
$$y(0) = 1,\ y'(0) = 1$$

$$2y'' - 3y' = 0$$
$$y(0) = 0,\ y'(0) = 0$$

ANSWERS [1]

[1] 1. $y(x) = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$ 2. $y(x) = \frac{3}{2}e^{\frac{2}{3}x} - \frac{3}{2}$

ASIDE:

Here are some ways to find a matrix inverse:

- By Hand:
  Given **A** the inverse is defined as
  $$\frac{1}{det(\mathbf{A})}\left[adj(\mathbf{A})\right]$$
  where $det(\mathbf{A})$ is the determinate and $adj($**A** is the adjoint. We can define the adjoint as the transpose of the cofactor matrix.

- Using Matlab or Freemat:
  Open up either Matlab or Freemat and in the command window type:

  ```
  A(1,:)= [1,1,1]
  A(2,:)=[0,-5,2]
  A(3,:)=[0,25,4]
  b=[7,0,70]
  b=transpose(b)
  C=inv(A)*b
  ```

  You should type these commands EXACTLY as you see them here for our example problem. Don't leave out the colons or the commas! The result should be a vector C that contains your answer.

The Moral(s) of the Story

- An n^{th} order Linear Homogeneous ODE has exactly n linearly independent solutions, assuming it meets the requirements for existence and uniqueness.

- We can assume solutions of the form $y = e^{rt}$ and find a characteristic equation, which allows us to solve for the values of r. For n^{th} order we would write our general solution as
  $$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

- You should try to review solving linear systems! Every time we apply our initial conditions we will be solving a system of equations. Come get help if you need it!