

Differential Equations – Notes

Professor:

Dr. Joanna Bieri

joanna_bieri@redlands.edu

Office Hours:

Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations – Day 13

We are continuing our discussion of linear higher order equations. Last time we found a solution method for linear constant coefficient equations. We will continue that discussion here.

Linear Constant Coefficient Equations – Solution Method

Recall from last time

1. We start by assuming a solution of the form $y(x) = e^{rx}$ and plugging this in gives us the characteristic equation. For second order this looks like

$$ar^2 + br + c = 0$$

2. We solve the characteristic equation for the roots and construct the solution using the Ansatz. For second order this looks like

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

3. After writing down the general solution we can apply the initial conditions, if given.
4. There are three cases for these roots: Real Distinct, Real Repeated, and Complex.

Repeated Roots

Let's see what happens when we have Repeated Roots in the Characteristic Equation.

EXAMPLE:

$$y^{(4)} + 4y^{(3)} = 0$$

Before we start let's answer some questions:

- How many linearly independent solutions do we expect?
- Do we expect for unique solutions to exist for all values of x ?

We should have four linearly independent solutions and a unique solution since we have constant coefficients.

First find the characteristic equation

$$r^4 + 4r^3 = 0$$

we can factor out r^3 and see that we have a triple repeated root at $r = 0$ and a single root at $r = -4$. So we would write the solution, using the methods we have so far, as

$$y(x) = c_1 e^{-4x} + c_2 e^0 + c_3 e^0 + c_4 e^0 = c_1 e^{-4x} + A$$

What is the problem here? That's right! We really only found two linearly independent solutions! There are two more solutions to this ODE that are "hidden" in that triple root! We need to find a way to construct linearly independent solutions!

Let's consider a simpler example to start

EXAMPLE:

$$y'' + 2y' + y = 0$$

The characteristic equation is $r^2 + 2r + 1 = (r + 1)^2 = 0$ so we have a double root at $r_1 = r_2 = -1$. This gives us ONE linearly independent solution $y_1 = e^{-x}$. Now using the substitution from last night's homework, we know one solution let's look for another solution in the form

$$y_2 = u(x)y_1$$

Taking derivatives of this yields

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$$

and plugging into our ODE we find

$$u'' y_1 + 2u' y_1' + u y_1'' + 2u' y_1 + 2u y_1' + u y_1 = 0$$

We can gather like derivatives of u to get

$$u'' y_1 + (2y_1' + 2y_1)u' + (y_1'' + 2y_1' + y_1)u = 0$$

But if y_1 satisfies the ODE then $y_1'' + 2y_1' + y_1 = 0$ and we are left with

$$u'' + (2y_1' + 2y_1)u' = 0$$

For us $y_1 = e^{-x}$ so $(2y_1' + 2y_1) = 0$ and our ODE becomes

$$u'' e^{-x} = 0 \rightarrow u'' = 0$$

this can be integrated twice to get $u = Ax + B$ so subbing into our assumption for y_2 we find

$$y_2 = (Ax + B)e^{-x}$$

giving us xe^{-x} as another linearly independent solution! Our general solution would be

$$y(x) = c_1e^{-x} + c_2xe^{-x}$$

You can imagine that for higher order equations we would just get higher order polynomials out of this type of substitution.

IN GENERAL - REPEATED ROOTS

Given a root r_1 to the characteristic equation with multiplicity k the part of the solution corresponding to r_1 is written

$$(c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1})e^{r_1x}$$

Now we can go back and finish up our original example:

EXAMPLE:

Recall our ODE

$$y^{(4)} + 4y^{(3)} = 0$$

had the roots $r_1 = 0$ with multiplicity three, triple repeated, and $r_2 = -4$. We can construct our solution using the general form above:

$$y(x) = (c_1 + c_2x + c_3x^2) + c_4e^{-4x}$$

Now we have four linearly independent solutions!

YOU TRY:

Construct a general solution given the following characteristic equations

$$1. (r + 1)^4(r - 1)(r - 2) = 0$$

$$2. (r - 2)^4(r - 1)^3(r + 1) = 0$$

ANSWERS ^{1 2}

Find the general solution to the ODE

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

ANSWER ³

$$^1 y(x) = c_1e^x + c_2e^{2x} + (c_3 + c_4x + c_5x^2 + c_6x^3)e^{-x}$$

$$^2 y(x) = c_1e^{-x} + (c_2 + c_3x + c_4x^2)e^x + (c_5 + c_6x + c_7x^2 + c_8x^3)e^{2x}$$

$$^3 y(x) = (c_1 + c_2x + c_3x^2) + (c_4 + c_5x)e^{\frac{x}{3}}$$

Complex Roots

Now we can consider our last case for roots to the characteristic equation, complex roots. We will start simple by first considering the case of purely imaginary roots.

EXAMPLE:

$$y'' + y = 0$$

The characteristic equation is $r^2 + 1 = 0$ giving roots $r = \pm\sqrt{-1} = \pm i$. So we could write our solution as

$$y(x) = c_1e^{ix} + c_2e^{-ix}$$

However this is not the ideal form for the solution. We expect an ode with real values coefficients and real valued initial conditions to have real valued solutions. We can use Euler's formula to rewrite our general solution

$$e^{ix} = \cos x + i \sin x$$

ASIDE: Euler's Formula Where does this formula come from? It is often introduced either in Calculus or in Complex analysis. Here is how we can prove to ourselves that it works. First consider the Taylor series for the function e^t

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

For our formula $t = ix$ and we recall that $i = \sqrt{-1}$ and $i^2 = -1$, so plugging in gives

$$e^{ix} = 1 + (ix) - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots$$

Now I can gather separately real and imaginary terms

$$e^{ix} = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots)$$

But wait a second here, these are just the Taylor series for Sine and Cosine! So we can write

$$e^{ix} = \cos x + i \sin x$$

We can use this to rewrite our general solution in our example

$$y(x) = c_1e^{ix} + c_2e^{-ix} = c_1(\cos x + i \sin x) + c_2(\cos x - i \sin x)$$

rearrange by gathering sin and cos

$$y(x) = (c_1 + c_2) \cos x + (c_1 - c_2)i \sin x = A \cos(x) + B \sin(x)$$

IN GENERAL - COMPLEX ROOTS

Given complex conjugate roots to the characteristic equation in the form $r = a \pm ib$ we write the corresponding part of the general solution as

$$y(x) = e^a(A \sin(bx) + B \cos(bx))$$

We can see this is true by checking the original solution form

$$y(x) = c_1 e^{a+ib} + c_2 e^{a-ib} = c_1 e^a e^{ib} + c_2 e^a e^{-ib}$$

We can factor out the real part leaving us with the imaginary part that can be rewritten using Euler's formula.

NOTE: Often in physics is to convenient to write the solution for imaginary roots as

$$y(x) = A \sin(bx + \psi)$$

where A is an amplitude and ψ is a phase shift. This is exactly the same as our solution just written a different way. We could use the identity

$$\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$$

to get between the different forms. When you write your solution with a phase shift notice that applying the initial conditions gives you A and ψ .

YOU TRY:

$$y'' - 4y + 5y = 0, \quad y(0) = 1, \quad y'(0) = 5$$

ANSWER ⁴

What would the general form solution be if the roots to the Characteristic Equation were $r = 0$ with multiplicity two and $r = 3 \pm 2i$ with multiplicity two. This means that really you have the following six roots

$$r = 0, 0, 3 \pm 2i, 3 \pm 2i$$

HINT: Apply the same patterns!

ANSWER ⁵

EXAMPLE:

Here is another fully worked example

$$y^3 + y' - 10y = 0$$

This gives the characteristic equation $r^3 + r - 10 = 0$.

Recall how to factor a cubic! We first work to guess a root using factors of the first and last term in this case 1, 2, 5, 10. Then guess a root, I will guess that $r = 1$ and plug in to see $(1)^3 + (1) - 10 \neq 0$ we need something to balance that -10 try $r = 2$ and plug in to see $(2)^3 + 2 - 10 = 8 + 2 - 10 = 0$ so $r = 2$ is a root. We can either use synthetic division to factor out the root or polynomial long division to divide out $(r - 2)$. Doing this gives $(r - 2)(r^2 + 2r + 5) = 0$

Next we solve the quadratic $(r^2 + 2r + 5) = 0$ using the quadratic formula $r = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2(1)}$ which gives us the complex roots $r = -1 \pm 2i$.

$$^4 y(x) = e^{2x}(\cos x + 3 \sin x)$$

$$^5 y(x) = c_1 + c_2 x + e^{2x}(c_3 + c_4 x)(c_5 \sin(3x) + c_6 \cos(3x))$$

$$^6 1. y(x) = c_1 e^{-2x} + (c_2 + c_3 x)e^{-3x}$$

$$^7 2. y(x) = e^{3x}(c_1 \sin(2x) + c_2 \cos(2x))$$

We construct our solutions using the rules for a distinct real root and a pair of complex conjugate roots

$$y(x) = c_1 e^{2x} + e^{-x}(c_1 \sin(2x) + c_3 \cos(2x))$$

YOU TRY:

$$1. y^{(3)} + 8y'' + 21y + 18 = 0$$

$$2. y'' - 6y + 13 = 0$$

ANSWER ^{6 7}

The Moral(s) of the Story

- Review how to find roots to cubic equations if you are having trouble with this. Also review basic ideas behind complex numbers like $a \pm bi$.
- It is best to memorize the general forms! This makes solving this type of problem very easy because all you have to do is find the roots and construct the solution.
- Make sure you understand how we are doing all of these substitutions. You want to make sure you understand all of the homework problems involving substitutions. These are not going away so please work to understand them.