Differential Equations - Notes

Professor:
Dr. Joanna Bieri
joanna_bieri@redlands.edu

Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 14
Take a minute and review what we have so far for linear constant coefficient equations. We are building directly on this foundation.

Nonhomogeneous Linear Constant Coefficient Equations

We are considering equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

We can use some nice shorthand notation here if we define the operator

$$L = a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 \frac{d}{dx} + a_0$$

then we write our linear system as

$$Ly = f(x)$$

If $f(x) = 0$ then we have our homogeneous case $Ly = 0$ and we do the following

1. Write down the characteristic equation
2. Find the roots of the characteristic equation
3. Write down the solution as the sum of solutions based on the roots. Here we could have Real Distinct, Repeated, or Complex Roots and each has a specific basic form.

Now we will consider $Ly = f(x)$ and we use a method that we developed a bit in past homework. We separate our work into two separate jobs:

1. We find solutions to the associated homogeneous problem $Ly_h = 0$
2. We find one solution that satisfies the nonhomogeneous problem $Ly_p = 0$

Then our general solution is the sum of these two $y = y_h + y_p$ where y_h will contain our unknown constants. We already know how to find y_h lets figure out how to find y_p.

NOTE: Sometimes y_h is called the "Complimentary Solution" and y_p is called the "Particular Solution".

Method of Undetermined Coefficients

The most basic, and most useful, method for finding the nonhomogeneous solution is called the Method of Undetermined Coefficients. The idea is that we make a general educated guess for what y_p could be and then we sub our guess into the nonhomogeneous ODE to figure out what the coefficients should be. Here is how we guess:

- If $f(x)$ is a polynomial of degree m then we guess a general polynomial containing all terms of degree m or lower.
- If $f(x)$ contains $\sin(ax)$ or $\cos(ax)$ we guess a linear combination of $\sin(ax)$ and $\cos(ax)$ with constants A and B
- If $f(x)$ contains e^{ax} then we guess Ae^{ax}
- There is a special case where our non-homogeneous term $f(x)$ repeats, or matches, our homogeneous solution. More on that later.

Here is an example

EXAMPLE:

$$y'' + 3y' + 4y = 3x + 2$$

1. We first solve the associated homogeneous problem.

   $$y'' + 3y' + 4y = 0$$

   Here is just pretend $f(x) = 0$ and solve like normal! The characteristic equation is $r^2 + 3r + 4 = 0$ and using the quadratic equation I find

   $$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 4(1)(4)}}{2} = \frac{-3}{2} \pm \frac{\sqrt{7}i}{2}$$

   so my homogeneous solution is

   $$y_h = e^{\frac{-3x}{2}} \left(A \sin\left(\frac{\sqrt{7}x}{2}\right) + B \cos\left(\frac{\sqrt{7}x}{2}\right)\right)$$

2. Next we solve for a nonhomogeneous part using **The method of undetermined coefficients**

   (a) Look at $f(x)$:
   $$f(x) = 3x + 2$$
   (b) Guess that y_p is similar in form to $f(x)$ and use the "rules" above. Here I guess
   $$y_p = ax + b$$
   (c) Plug your guess into the ODE
   $$y_p' = a, \quad y_p'' = 0$$
   $$0 + 3a + 4ax + 4b = 3x + 2$$
   $$(4a)x + (3a + 4b) = 3x + 2$$
   and solve for the constants that will balance this equation.
   $$4a = 3, \quad a = \frac{3}{4}$$
   $$3a + 4b = 3\frac{3}{4} + 4b = \frac{9}{4} + 4b = 0, \quad b = \frac{-1}{16}$$
   (d) Your nonhomogeneous solution is given by
   $$y_p = \frac{3x}{4} - \frac{1}{16}$$

3. Now we write down our full general solution to the ODE as the sum of the two pieces $y = y_h + y_p$ or in this case
$$y(x) = e^{\frac{-3x}{2}}\left(A\sin\left(\frac{\sqrt{7}x}{2}\right) + B\cos\left(\frac{\sqrt{7}x}{2}\right)\right) + \frac{3x}{4} - \frac{1}{16}$$

EXAMPLE:
$$y'' - 4y = 2e^{3x}$$

1. We first solve the associated homogeneous problem.
$$y'' - 4y = 0$$
Our characteristic equation is $r^2 - 4 = 0$ so we have roots $r = \pm 2$ and our homogeneous solution is
$$y_h = c_1e^{2x} + c_2e^{-2x}$$

2. Next we solve for a nonhomogeneous part using **The method of undetermined coefficients**

   (a) Look at $f(x)$:
   $$f(x) = 2e^{3x}$$
   (b) Guess that y_p is similar in form to $f(x)$ and use the "rules" above. Here I guess
   $$y_p = ae^{3x}$$
   (c) Plug your guess into the ODE
   $$y' = 3ae^{3x}, \quad y'' = 9ae^{3x}$$
   $$9ae^{3x} - 4ae^{3x} = 5ae^{3x} = 2e^{3x}$$
   and solve for the constants that will balance this equation.
   $$5a = 2, \quad a = \frac{2}{5}$$
   (d) Write down the nonhomogeneous solution
   $$y_p = \frac{2}{5}e^{3x}$$

3. Now we write down our full general solution to the ODE as the sum of the two pieces $y = y_h + y_p$ or in this case
$$y(x) = c_1e^{2x} + c_2e^{-2x} + \frac{2}{5}e^{3x}$$

IMPORTANT NOTE: You would only apply initial conditions AFTER writing down the full general solution $y = y_h + y_p$. Your nonhomogeneous part will effect your constants!

YOU TRY:
$$y'' - y = \sin(2x)$$

ANSWER [1]
Lets get some practice guessing the particular solution.

EXAMPLE:
What would you choose for a particular solution, given $f(x)$, in each of the following cases:

A
$$f(x) = x^2$$
Here I would guess a general degree two polynomial
$$y_p = ax^2 + bx + c$$
Notice how I include all of the lower order terms here!

B
$$f(x) = e^x + \sin(x) + x$$
For this problem you need to account for each of the different types of functions. In fact you could do them separately, treating $f(x)$ as $f_1(x) + f_2(x) + f_3(x)$ and finding a nonhomogeneous term for each of the three terms. A general guess would look like
$$y_p = ae^x + b\sin(x) + c\cos(x) + dx + e$$

[1] $y(x) = Ae^{-1} + Be^x - \frac{1}{3}\sin(2x)$

C

$$f(x) = 4x^3 + x$$

Here I don't need to make a separate guess for each term since they will both be included in the guess for the third order polynomial.

$$y_p = ax^2 + bx^2 + cx = d$$

Method of Undetermined Coefficients - Duplication

Let's consider a case where we get into trouble with our method of undetermined coefficients

EXAMPLE:

$$y'' - 4y = 2e^{2x}$$

We solved a problem similar to this earlier, the only difference is in the nonhomogeneous part we changed the exponent from $3x$ to $2x$. From before we have

$$y_h = Ae^{2x} + Be^{-2x}$$

Now normally we would pick our nonhomogeneous term using undetermined coefficients by looking at $f(x)$ and we would get

$$y_p = ae^{2x}$$

But what happens when we plug this in? That's right! It disappears because IT IS THE HOMOGENEOUS SOLUTION! Whenever our nonhomogeneous term is a copy of the homogeneous solution we will have this problem. We can use the same trick as before when we had repeated roots and search for solutions of the form $y_p = v(x)e^{2x}$. Doing this we would find that when we have repeats we multiply our guess by an extra x to get a linearly independent solution that is not a copy. In other words, a better guess for the nonhomogeneous solution would have been

$$y_p = axe^{2x}$$

If we plug this in we see we can solve for $a = \frac{1}{2}$ and putting it all back together we would find

$$y = Ae^{2x} + Be^{-2x} + \frac{1}{2}xe^{2x}$$

as our general solution.

YOU TRY:

What should we guess for the form of the nonhomogeneous solution for the following problems?

A

$$y'' + 6y' + 13y = e^{-3x}\cos(2x) + x$$

ANSWER [2]

B

$$Ly = x^2e^{2x} + x\sin(3x)$$

where we are given the characteristic equation

$$(r-2)^3(r^2+9)$$

ANSWER [3]

COMMENT:

Here I want to make a quick comment about when the Method of Undetermined Coefficients won't work or when we run into trouble. Notice we need a nice form for $f(x)$ in order to be able to guess y_p and in general the guess matches y_p and accounts for all possible derivatives that might match. MUD breaks down when you can't write down all of the derivatives nicely. Here are some examples:

- When $f(x) = \frac{1}{x^n}$ for $n > 0$. Notice that if I start taking derivatives of terms that look like $f(x)$ I would never be able to stop taking derivatives. In other words $y_p = \frac{a}{x^n} + \frac{b}{x^{n+1}} + \frac{c}{x^{n+2}} + \cdots$ It would be never ending and I would not have something nice to plug in.
- When $f(x) = \tan x$ again we get endless derivatives and can't make a nice guess!
- For functions like $\sin^2(x)$ we can't directly guess this one, but this is a case where trigonometric identities can help you to rewrite $f(x)$ in a better form for making your guess!!

The Moral(s) of the Story

- We can use the method of undetermined coefficients on any Linear Constant Coefficient ODE, $Ly = f(x)$ as long as $f(x)$ is sufficiently "nice", meaning that we can easily guess the form of the nonhomogeneous solution.
- The basic method goes as follows
  1. Solve $Ly_h = 0$ for the homogeneous solution.
  2. Guess y_p by looking at $f(x)$ and checking for duplicates between your guess and y_h.
  3. Sub your guess for y_p back into $Ly = f(x)$ and solve for the constants.
  4. Write your final solution as $y = y_h + y_p$
  5. Apply your initial conditions only after you have the FULL general solution.

[2] $y_h = e^{3x}(A\sin(2x) + B\cos(2x))$ and this duplicates the nonhomogeneous term so our best guess is $y_p = xe^{3x}(a\sin(2x) + b\cos(2x)) + cx + d$

[3] $y_h = (c_1 + c_2x + c_3x^2)e^{2x} + c_4\sin(3x) + c_5\cos(3x)$ here we see duplications everywhere! Our best guess is $y_p = x^3(a + bx + cx^2)e^{2x} + x[(d + ex)\sin(3x) + (f + gx)\cos(3x)]$ Notice here how we multiplied the leading polynomial by x^3 to remove ALL duplications.