Differential Equations - Notes

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 15
Today we will do one more example for Method of Undetermined Coefficients and I will share some SageMath code to help you simplify your equations when solving for these constants. I know this gets complicated, but be patient and keep trying!

Method Of Undetermined Coefficients - Using SageMath

SageMath is a freely available mathematics program that you can either run on a web browser or download and install on your machine. I would suggest that you try to install it, since it will run much faster on your computer than online. Either way you can find the program at

http://www.sagemath.org/

We will see how to use this with an example.

EXAMPLE:

$$y'' + 2y' + y = \sin(x) + 3e^{-x}$$

First we always solve the associated homogeneous problem.

Homogeneous Part:
The characteristic equation is $r^2 + 2r + 1 = 0$ so we find $(r+1)^2 = 0$ or a double root at $r = -1$ giving the homogeneous solution as

$$y_c = (c_1 + c_2 x)e^{-x}$$

Nonhomogeneous Part:
We are going to use MUC so we need to make a guess as to the form of the nonhomogeneous solution y_p. We guess it to match the right hand side of our equation: $f(x) = \sin(x) + 3e^{-x}$. Our naive guess is

$$y_p = a\sin(x) + b\cos(x) + ce^{-x}$$

but now we double check to see if any part of our guess matches our homogeneous solution. It does the ce^{-x} part matches our y_c. So we improve our guess to get rid of the duplication

$$y_p = a\sin(x) + b\cos(x) + cx^2e^{-x}$$

Notice here that I needed to multiply just the last term by x^2 in order to get rid of all duplications. Now we can continue with our method

1. Plug our guess y_p into the left hand side of the ODE
2. Simplify and gather like terms in $\sin(x)$, $\cos(x)$, and e^{-x}
3. Choose the parameters a, b, and c to balance two sides of the equation.

We can definitely do this by hand, but here is an example of how to do it on SageMath.

First create a new empty Sage Worksheet. This should look like an empty page with some buttons at the top. You type your commands, or equations, into the blank space. Any line that starts with a # is a comment and just there to give us hints about how to use the code. Here is the program that we are going to use:

```
# This worksheet helps with taking derivatives and simplifying for The Method of Undetermined Coeficients

# You will need to make changes to each section
# You still need to use your brain to make this work, look at each output and make sure it looks correct!
# Press CONTROL-ENTER after each section to break it into cells. Then push Run to evaluate

# Define your constants and  yp  based on your very good guess
# YOU WILL NEED TO CHANGE THESE TO MATCH YOUR GUESS
a=var('a')
b=var('b')
c=var('c')
yp(x)=a*sin(x)+b*cos(x)+c*x^2*e^(-x)
yp


# Take the derivatives
# NO NEED TO CHANGE
yp_p = derivative(yp,x)
yp_p

yp_pp = derivative(yp_p,x)
yp_pp


# Plug into the Left Hand Side of the ODE
# YOU NEED TO CHANGE THIS TO MATCH YOUR ODE
Ly = yp_pp+2*yp_p+yp
Ly


# Enter f(x)
# YOU NEED TO CHANGE THIS TO MATCH YOUR f(x)
f = sin(x) +3*e^(-x)
f


# Compare the coeficients
# NO NEED TO CHANGE
Ly2 = Ly.maxima_methods()
Ly2.collectterms(sin(x),cos(x),e^(-x))
Ly2


# NOW YOU COMPARE THE OUTPUT FOR Ly2 and f(x)
# This should give you n-equations for n-unknowns
# PLUG THE N-EQUATIONS for N-UNKNOWNS IN BELOW
solve([2*a==0, -2*b==1, 2*c==3 ], a,b,c)
```

You can copy and past the code above into a Sage Worksheet to see how it works for this problem. You should find that $a = 0, \quad b = \dfrac{-1}{2}, \quad c = 3/2$

We will work in class to build this program together and answer questions about how all of the pieces work together.

Now that you have a way to do this calculation on a computer you are welcome to use this on your homework. I still expect you to "show your work" which now consists of printing out your code for the given problem.

`Variation of Parameters`

Sometimes we cannot come up with a good guess for MUC. This can happen whenever $f(x)$ is not in one of our basic guessing forms $\sin(ax) + \cos(ax)$ or x^n or e^{ax}. In these cases we need to develop a new method for finding y_p. As it turns out Variation of Parameters is a method that can find y_p for any linear nonhomogeneous ODE. Here is the basic idea:

Given a general linear ODE

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + P_{n-2}(x)y^{(n-2)} + \cdots + P_1(x)y' + P_0 y = f(x)$$

- We always first find the homogeneous solution.
- Then assume the nonhomogeneous solution can be written in the form

  $$y_p = u_1(x)y_1 + u_2(x)y_2 + \cdots$$

  where $y_1, y_2, \cdots$ are the linearly independent homogeneous solutions.
- We want to plug this into our ODE so we start taking derivatives.
- BUT each time we take a derivative, we suppress (aka set equal to zero), the sum of any terms that would lead to higher than first derivatives in u.
- After plugging into the ODE and canceling homogeneous terms, we should be left with a system of n-equations for the each of the u_n'.
- Solving this system gives us a formula for Variation of Parameters.

As you can imagine, for higher order, this gets very messy. So today we will derive the method for second order equations.

$$y'' + p(x)y' + q(x)y = f(x)$$

Notice that I am not putting any restrictions on $p(x)$ and $q(x)$ other than that solutions exist. This means that our formula for Variation of Parameters will work on any type of second order equation, not just constant coefficient. Okay, here we go with the formulation:

Assume that we have a homogeneous solution to the second order equation

$$y_c = c_1 y_1 + c_2 y_2$$

since this is usually easy to find. Now we will assume that our nonhomogeneous solution is of the form

$$y_p = u_1 y_1 + u_2 y_2$$

where u_1 and u_2 are some unknown functions of x. Let's take derivatives!

$$y + p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

now looking at this equation, if I took another derivative, an term that contains a u' would lead to higher derivatives so I assume

$$u_1' y_1 + u_2' y_2 = 0$$

to suppress those derivatives. Leaving

$$y_p' = u_1 y_1' + u_2 y_2'$$

taking another derivative I find

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_1' y_1' + u_1 y_2''$$

Now we will plug y_p, y_p', and y_p'' into our ODE leading to a big messy looking formula... but gathering terms a bit leads to

$$u_1' y_1' + u_2' u_2' + u_1[y_1'' + p(x)y_1' + q(x)y_1] + u_2[y_2'' + p(x)y_2' + q(x)y_2] = f(x)$$

and since y_1 and y_2 are homogeneous solutions, most of this goes to zero! Leaving

$$u_1' y_1' + u_2' u_2' = f(x)$$

So between this equation and my assumption to suppress higher order derivatives I find two equations for my two unknowns u_1' and u_2'

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' u_2' = f(x)$$

in matrix form this becomes

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

We can solve this using the matrix inverse

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{det\left(\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\right)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

But that determinant in the denominator should look familiar! It is the Wronskian so we can write

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f(x) \end{bmatrix} = \begin{bmatrix} \frac{-1}{W} y_2 f(x) \\ \frac{1}{W} y_1 f(x) \end{bmatrix}$$

So we have two ODE that we can simply integrate to solve for u_1 and u_2

$$u_1 = \int \frac{-1}{W} y_2 f(x)\, dx$$

$$u_2 = \int \frac{1}{W} y_1 f(x)\, dx$$

Giving us a formula for our particular solution

$$y_p = -y_1 \int \frac{y_2 f(x)}{W}\, dx + y_2 \int \frac{y_1 f(x)}{W}\, dx$$

This is a formula that you can memorize and use for Variation of Parameters in second order linear equations.

CHALLENGE: See if you can derive the formula for a third order equation! What about in general for an n^{th} order linear ODE?

EXAMPLE:
Solve using Variation of Parameters

$$y'' + 3y' + 2y = 4e^x$$

First we always need to find the solution to the associated homogeneous problem. Here we see our characteristic equation is $r^2 + 3r + 2 = 0$ which gives $r = -1, -2$ so

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

Now we need to choose one of these to be y_1 and the other to be y_2. It doesn't matter how we pick, but once we choose them they must remain the same for the rest of the problem. I will pick

$$y_1 = e^{-x} \quad y_2 = e^{-2x}$$

then I need to compute the Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

Next we just plug into the formula and evaluate the integrals. I like to separate this into two parts

$$y_p = I_1 + I_2$$

where

$$I_1 = -y_1 \int \frac{y_2 f(x)}{W}\, dx$$

and

$$I_2 = y_2 \int \frac{y_1 f(x)}{W}\, dx$$

Let's do the first integral

$$I_1 = -e^{-x} \int \frac{e^{-2x} 4e^x}{-e^{-3x}}\, dx = 4e^{-x} \int e^{2x}\, dx = 2e^x$$

We can do a similar calculation to find that

$$I_2 = \frac{-4}{3} e^x$$

so

$$y_p = I_1 + I_2 = \frac{2}{3} e^x$$

and we would write our full solution as the sum of the homogeneous and the particular.

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{2}{3} e^x$$

The nice thing about Variation of Parameters is that we can solve equations that have non constant coefficients. For example Euler Equations that are nonhomogeneous can be solved by first finding the homogeneous solution using the sum $v = ln(x)$ like we have done in the homework and then using variation of parameters. Here is one to try.

YOU TRY:

$$x^2 y'' - 3xy' + ry = 3x$$

Were you are given $y_c = c_1 x^2 + c_2 \ln(x) x^2$ First you will want to divide the ODE by x^2 to get it in the correct general form and giving you the correct $f(x)$! ANSWER [1]

`The Moral(s) of the Story`

- We now have computer programs that can help us with Method Of Undetermined Coefficients. BUT - you must still use your brain to find a good guess for y_p and figure out your equations for the unknown coefficients.
- Method of Undetermined Coefficients only works for linear constant coefficient equations!
- Variation of Parameters will work for any linear nonhomogeneous ODE. We developed the formula for second order equations.
- To use the formula you need to find the homogeneous solution, y_1 and y_2, then calculate the Wronskian, and finally sub everything in and solve those integrals.

[1] $y = c_1 x^2 + c_2 \ln(x) x^2 + 3x^3$