Differential Equations - Notes

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 17
Today we shift our attention to a slightly different type of problem. Instead of being given initial conditions, we will consider the case of boundary conditions.

Endpoint problems

So far we have talked a lot about Initial Value Problems (IVP)

$$y'' + p(x)y' + q(x)y = f(x)$$

GIVEN:

$$y(a) = b_1, \quad y'(a) = b_2$$

Here we are given information about the value and the slope at one point. In general IVPs have nice solutions because it is fairly easy to apply conditions at a single point.

Today we will change our focus to talk about Boundary Value Problems (BVP). The only change is in the extra conditions given.

$$y'' + p(x)y' + q(x)y = f(x)$$

GIVEN:

$$y(a) = b_1, \quad y(c) = b_2$$

These are actually much more complicated and more often than not, don't have a solution! Why? Well think about the types of functions that we have been getting as solutions, for example $y = e^x$. Once we pick a location that it goes through on one end $y(a) = b_1$ there is one and only one point that it can go through at $x = c$. So if we didn't happen to perfectly pick $y(c) = b_2$ then we won't get a solution.

EXAMPLE:

$$y'' + 4y = 0$$

$$y(0) = 0, \quad y(\pi) = 0$$

This is just a linear constant coefficient homogeneous ODE so we solve it just like normal! Consider the characteristic equation $r^2 + 4 = 0$ then we find $r = \pm 2i$ so

$$y(x) = A\sin(2x) + B\cos(2x)$$

Now let's apply the given conditions. First $y(0) = 0$ so

$$A\sin(0) + B\cos(0) = B = 0$$

then

$$y(\pi) = A\sin(2\pi) = 0 = 0$$

Because $\sin(0) = \sin(2\pi) = 0$ we are never able to find a number for the constant A. This means solutions are not unique and any function $y = A\sin(x)$ satisfies the BVP.

EXAMPLE:

$$y'' + 4y = 0$$

$$y(0) = 0, \quad y(1) = 0$$

This is the same problem again, I just changed on of the boundary conditions! So we have

$$y = A\sin(2x) + B\cos(2x)$$

and applying the first condition still gives $B = 0$. Now let's apply the second condition

$$y(1) = A\sin(2) = 0$$

well this means that $A = 0$ and our only solution is the trivial solution $y = 0$. Actually for most boundary conditions we would pick, this BVP would only have a trivial solution

BVPs usually have these types of problems!! More often than not they have trivial solutions. So what we do is ask a different mathematical question! **Given a set of boundary conditions, which ODEs would have a non-trivial solution?** This leads us to the idea of an Eigenvalue Problem

Eigenvalue Problem

The general form that we will consider for our Eigenvalue Problems is

$$y'' + p(x)y' + \lambda q(x)y = 0$$
$$y(a) = 0, \quad y(c) = 0$$

Notice here that we are only looking at homogeneous problems with homogeneous boundary conditions. These do get more complicated that what we will see in this class! The new parameter λ is called the eigenvalue. Part of our goal in solving these types of problems is to find the eigenvalue(s). Let's jump in a look at an example.

EXAMPLE:

$$y'' + \lambda y = 0$$
$$y(0) = 0, \quad y(L) = 0$$

Where we assume $L > 0$. Let's work through the solution to this second linear homogeneous constant coefficient BVP.

First we look for a general solution by solving the characteristic equation

$$r^2 + \lambda = 0, \quad r = \pm\sqrt{\lambda}$$

So how we write our general solution depends on the values of lambda! What are all the possibilities? If $\lambda < 0$ then we get real distinct roots, if $\lambda = 0$ then we get real repeated roots, and if $\lambda > 0$ then we get imaginary roots. We can't rule any of these cases out at this point so we will consider each of them independently:

A. $\lambda < 0$
Here we will let $\lambda = -a^2$ to force $\lambda < 0$ and make the square root go away. We will undo this later. With this we find $r = \pm a$ and get our general solution

$$y = c_1 e^{ax} + c_2 e^{-ax}$$

Now we will apply the boundary conditions and see what happens:

$$y(0) = c_1 + c_2 = 0, \quad \text{so} \quad c_1 = -c_2$$
$$y(L) = c_1 e^{aL} + c_2 e^{-aL} = 0, \quad \text{so} \quad c_1 = -c_2 e^{-2aL}$$
$$c_2 = c_2 e^{-2aL}$$

This means that either $e^{-2aL} = 1$ meaning that $a = 0$ which is not allowed because we already said that $\lambda < 0$ not equal zero, or $c_2 = 0$, which means $c_1 = 0$ and our only solution is the trivial one $y = 0$.

So our conclusion is that we cannot find nontrivial solutions to this BVP for the case of $\lambda < 0$!

B. $\lambda = 0$
Here we will let $\lambda = 0$ and we find $r = 0$ and get our general solution

$$y = c_1 + c_2 x$$

Now we will apply the boundary conditions and see what happens:

$$y(0) = c_1 = 0$$
$$y(L) = c_2 L = 0$$

This means that either $L = 0$ meaning we strangely join up our boundaries, aka we shrink our domain down to zero, no thanks, or $c_2 = 0$ and again our only solution is the trivial one $y = 0$

So our conclusion is that we cannot find nontrivial solutions to this BVP for the case of $\lambda = 0$!

C. $\lambda > 0$
Here we will let $\lambda = a^2$ and we find $r = \pm ai$ and get our general solution

$$y = c_1 \sin(ax) + c_2 \cos(ax)$$

Now we will apply the boundary conditions and see what happens:

$$y(0) = c_1 \sin(0) + c_2 \cos(0) = c_2 = 0$$
$$y(L) = c_1 \sin(aL) = 0$$

This means that either $c_2 = 0$ meaning that again our only solution is the trivial one $y = 0$, or $\sin(aL) = 0$. BUT WAIT! This is okay! We can allow $\sin(aL) = 0$. This actually works for a lot of cases, anytime $aL = n\pi$. We just found values of λ that would give unique solutions! We get nice solutions anytime $a = \frac{n\pi}{L}$ or when $\lambda = \left(\frac{n\pi}{L}\right)^2$.

This means that nontrivial solutions exist and we say solutions are given by

EIGENVALUES

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

with the associated

EIGENFUNCTIONS

$$y_n = \sin\left(\frac{n\pi x}{L}\right)$$

What does this solution mean? How can this possibly apply to the real world?

Real World Eigenvalue Problem

The equation for a vibrating or whirling string between two fixed points is given by

$$Ty'' + \rho\omega^2 y = 0$$

where T is the tension of the string, ρ is the linear density, and ω is the angular speed. Imagine a jump rope being held and spun and a circle by two people, T is how tightly they are pulling against the rope, ρ is the material weight of the rope, and ω is how fast their hands are spinning. Okay, we can actually rewrite this equation to be in exactly the form of the equation we just solved

$$y'' + \frac{\rho\omega^2}{T} y = 0$$

and our boundary conditions just say that each person, one at $x = 0$ and the other at $x = L$, is holding the rope on the same level and we are calling that level our zero.

$$y(0) = 0, \quad y(L) = 0$$

Our solutions were

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$y_n = \sin\left(\frac{n\pi x}{L}\right)$$

So given a string, aka specifying a length L and a density ρ, and holding it between two people, aka specifying a tension T, our eigenvalues specify an angular speed

$$\frac{\rho\omega^2}{L} = \left(\frac{n\pi}{L}\right)^2$$

solving for ω

$$\omega = \sqrt{\frac{T}{\rho}}\frac{n\pi}{L}$$

So does this mean that only certain angular speeds result in a solution? Well, yes!

Think about the jump rope! Go get a jump rope and a friend if you can... because we can make this work in real life.

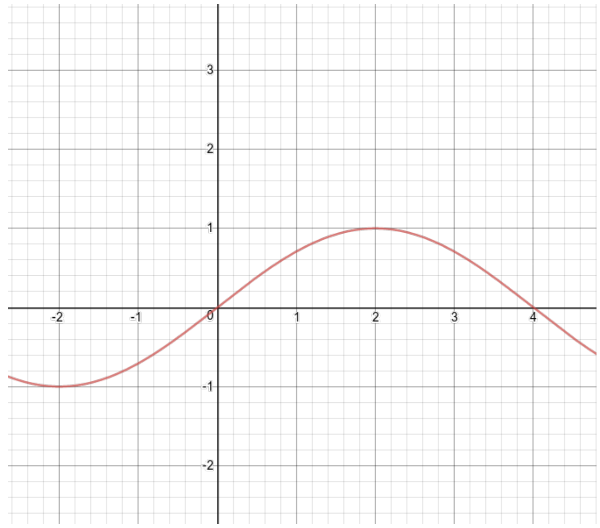
Now start spinning the rope like you would to jump. You see that you need to be spinning at a specific frequency to get the nice arcing rope. This frequency is

$$\omega_1 = \frac{\pi\sqrt{T}}{L\sqrt{\rho}}$$

and our associated function is

$$y_1 = \sin\left(\frac{\pi x}{L}\right)$$

Graphing this we see one arc fitting between $x = 0$ and $x = L$



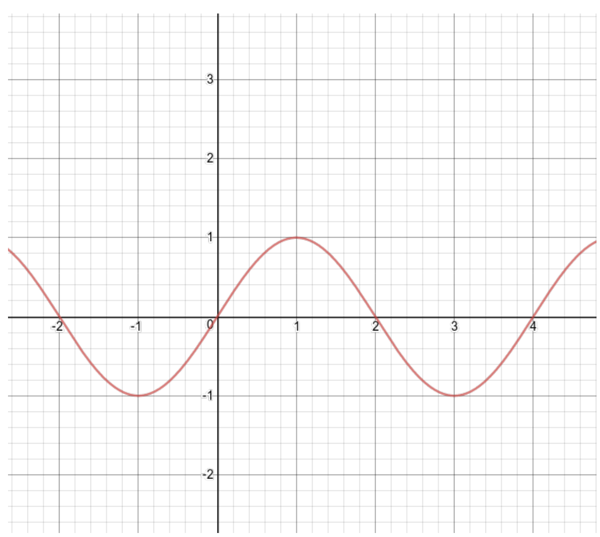
Now lets start spinning faster. What happens? The solution falls apart for a second and the rope looks like a mess until you hit on the next frequency and get two really nice arcs! The frequency is

$$\omega_1 = \frac{2\pi\sqrt{T}}{L\sqrt{\rho}}$$

and our associated function is

$$y_1 = \sin\left(\frac{2\pi x}{L}\right)$$

Graphing this we see two arcs fitting between $x = 0$ and $x = L$, where for the graph I set $L = 4$.



WOW! It really works!

We also see these types of things in Quantum Mechanics. The allowable energy levels of the wave function have to do with eigenvalues to the wave equation.

Back to Eigenvalue Problem

Let's work together to solve some slightly more complicated cases.

YOU TRY:

$$y'' + \lambda y = 0$$

Given

$$y(0) = 0, \quad y'(L) = 0$$

Here we have not changed the ODE we just changed the second boundary condition to a slope condition. Here are your steps:

1. Solve the characteristic equation for r [1]
2. Consider the possible cases for λ [2]
3. Apply the boundary conditions to each case and look for a nontrivial solution. [3]

In general to find your eigenvalues you end up solving a system of two equations for two unknowns

$$\alpha_1(\lambda)c_1 + \beta_1(\lambda)c_2 = 0$$

and

$$\alpha_2(\lambda)c_1 + \beta_2(\lambda)c_2 = 0$$

where here the α and β functions are what you get after you plug your boundary conditions in. For our first example this was $\alpha_1(\lambda) = \sin(0)$ and $\alpha_2(\lambda) = \sin(L)$ Writing this in matrix form

$$\begin{bmatrix} \alpha_1(\lambda) & \beta_1(\lambda) \\ \alpha_2(\lambda) & \beta_2(\lambda) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where at least one of c_1 or c_2 is nonzero. For this to be the case the determinant must be zero

$$\alpha_1(\lambda)\beta_2(\lambda) - \alpha_2(\lambda)\beta_1(\lambda) = 0$$

Seeking values of λ that solve this equation gives us our solutions.

In the last YOU TRY example we found

$$y = c_1 \sin(ax) + c_2 \cos(ax)$$

Applying the boundary conditions gives

$$c_1 \sin(0) + c_2 \cos(0) = 0$$

$$ac_2 \cos(aL) - ac_2 \sin(AL) = 0$$

from the matrix form above we can write this as

$$-a \sin(0) \sin(aL) - a \cos(0) \cos(aL) = 0$$

and simplifying we find

$$-a \cos(aL) = 0$$

which only gives nontrivial solutions when

$$aL = \frac{\pi}{2}, \frac{3\pi}{2}, \dots = \frac{(2n-1)\pi}{2L}$$

These problems can get much harder! In fact sometimes we can't even find a closed form solution for our eigenvalues. Here is a tough example.

[1] Show that you can do this and get $r = \pm\sqrt{-\lambda}$

[2] $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$

[3] Here you should find that $\lambda = \left(\frac{(2n-1)\pi}{2L}\right)^2$ and $y_n = \sin(\sqrt{\lambda}x)$

EXAMPLE:

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) + y'(1) = 0$$

Here we have a mixed boundary condition at $x = 1$. First the ODE is the same as before and we know that we will only have positive eigenvalues, meaning that we can write $\lambda = a^2$ and we have general solutions of

$$y = c_1 \sin(ax) + c_2 \cos(ax)$$

Now we apply the boundary conditions"

$$ac_1 \cos(0) - ac_2 \sin(0) = 0$$

$$c_1 \sin(a) + c_2 \cos(a) + ac_1 \cos(a) - a_c 2 \sin(a) = 0$$

gathering c_1 and c_2 gives

$$c_1(\sin(a) + a\cos(a)) + c_2(\cos(a) - a\sin(a)) = 0$$

Using the trick of the zero determinant we can write

$$a\cos(0)(\cos(a) - a\sin(a)) + a\sin(0)(\sin(a) + a\cos(a)) = 0$$

We need to simplify this!

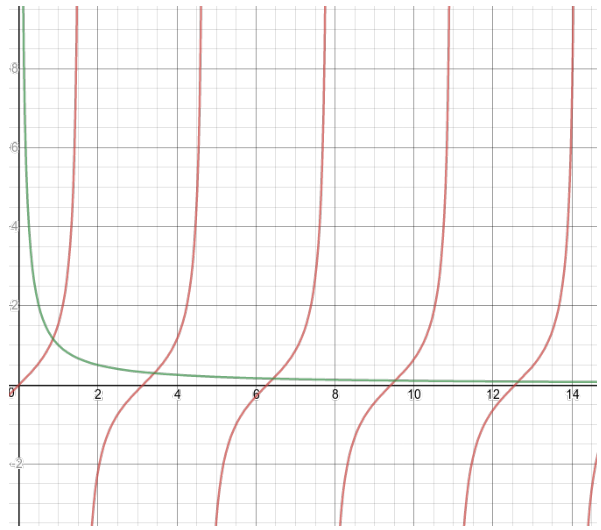
$$\begin{aligned} a\cos(a) - a^2\sin(a) &= 0 \\ \cos(a) - a\sin(a) &= 0 \\ \cos(a) &= a\sin(a) \\ \frac{1}{a} = \frac{\sin(a)}{\cos(a)} &= \tan(a) \end{aligned}$$

So our eigenvalues are given by the solution to

$$\frac{1}{a} = \tan(a)$$

we would have to solve for these numerically.

We can also see them on the graph as the intersection points between the line $y = \frac{1}{x}$ and $y = \tan(x)$.



This gives $\lambda \approx 0.86$, $\lambda \approx 3.426$, $\lambda \approx 6.437$...

The Moral(s) of the Story

- Boundary Value Problems (BVP) are fundamentally different than Initial Value Problems (IVP). Specifying the conditions on two boundaries makes our ODEs much less likely to have a nontrivial solution.
- When solving BVPs we often refocus our attention on the associated eigenvalue problem and ask the question "For what values of λ would our BVP have a nontrivial solution?"
- We always solve these problems by doing the following
  1. Solve the characteristic equation for r in terms of lambda and identify the possible cases for when r is real distinct, real repeated, or complex.
  2. Consider each of the cases above separately by writing down the general solution and applying the boundary conditions.
  3. Each time you apply the boundary conditions you are looking for possible values of λ, or if you set $\lambda = a^2$ then you look for values of a, that give you a nontrivial solution.
  4. You final solution is a set of EIGENVALUES and their associated EIGENFUNCTIONS.