Differential Equations - Notes

Professor:
Dr. Joanna Bieri
joanna_bieri@redlands.edu

Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 18
We have been focusing on how to solve non-homogeneous equations, but have not talked much about what all of this means physically. Today we consider applications for non homogeneous equations.

Mechanical Vibrations

Many physical systems can be modeled using the Damped Spring Mass Equation

$$mx'' + cx' + kx = f(t)$$

where $x(t)$ is the displacement from equilibrium at time t, m is the mass of the moving part, c is the damping coefficient, and k is the spring constant for the vibration. FYI - in physics we sometimes represent derivatives with dots over the x's, just in case you run into that notation it looks like

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

This equation can actually model a lot more than just our simple spring system and we will talk more about this later. For now, let's consider what these terms tell mean in the real world. We will break this up into two cases

- $f(x) = 0$ - FREE MOTION
- $f(x) \neq 0$ - FORCED MOTION

Case 1 - Free Motion - Homogeneous

Let's first put the equation in more standard form and divide by m

$$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$$

combining constants gives

$$x'' + px' + \omega x = 0$$

The characteristic equation for our homogeneous system is

$$r^2 + pr + \omega = 0$$

use the quadratic equation to find

$$r = \frac{-p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4\omega}$$

So solutions depend on the form of the thing under the square root!

1. $p^2 - 4\omega > 0$ or $p^2 > 4\omega$ we would expect real distinct roots. Physically this means that the damping force is greater than the spring force!

2. $p^2 - 4\omega = 0$ or $p^2 = 4\omega$ we would expect real repeated roots. Physically this means that the damping force is equal to the spring force.

3. $p^2 - 4\omega < 0$ or $p^2 < 4\omega$ we would expect complex roots. Physically this means that the damping force is less than the spring force.

Now we can thing about the solutions for each of these cases:

OVER DAMPED: $p^2 > 4\omega$

In this case we have real distinct roots so our solutions take the form

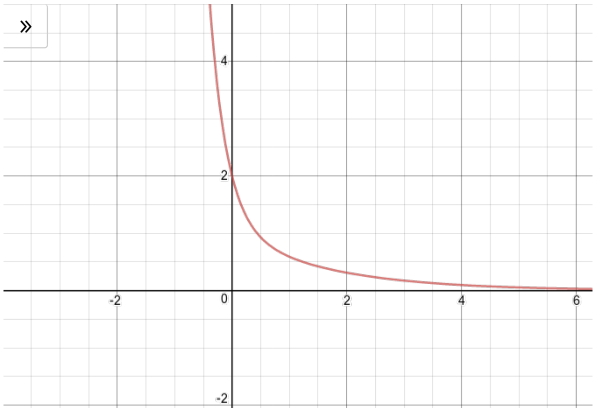
$$x(t) = c_1 e^{\frac{-p}{2}t + \frac{t}{2}\sqrt{p^2-4\omega}} + c_2 e^{\frac{-p}{2}t - \frac{t}{2}\sqrt{p^2-4\omega}t}$$

or

$$x(t) = \left[c_1 e^{\frac{t}{2}\sqrt{p^2-4\omega}} + c_2 e^{-\frac{t}{2}\sqrt{p^2-4\omega}t}\right] e^{-\frac{pt}{2}}$$

What do we expect from these solutions? Well notice how no mater what p is, we have that damping term in the leading negative exponential. This means that our solutions are decaying toward zero. How fast they go to zero depends on how much damping there is in the system. If p is very large, then the positive term $\frac{t}{2}\sqrt{p^2 - 4\omega}$ keeps the solution above zero for longer. Imagine in a physical system, if you are completely resisting motion then the spring with just slowly pull you back to equilibrium, but you will resist all the way!

Imagine a car with shocks that are too strong. You stand on the bumper and jump off, but rather than shifting right back into place, it SLOWLY returns to position. There will be no oscillations here. As an example we can graph the solution for $p = 4$ and $\omega = 1$

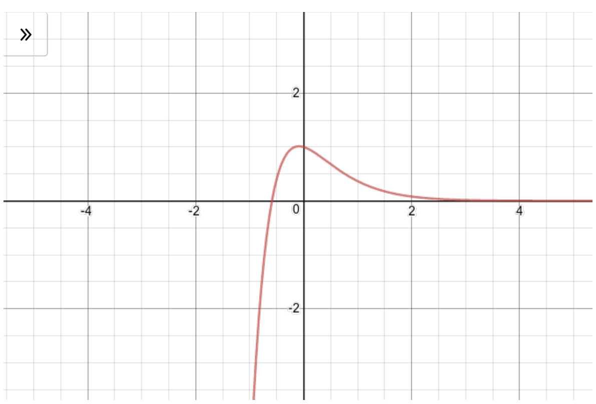


CRITICALLY DAMPED: $p^2 = 4\omega$

In this case we have real repeated roots so our solutions take the form

$$x(t) = (c_1 + c_2 t)e^{-\frac{pt}{2}}$$

What do we expect from these solutions? Only decay! The solutions should go quickly to zero when the oscillations are critically damped. For our car example, this is the best case scenario. You just off the bumper and the car quickly returns to equilibrium without oscillations. Here is an example graph for when $p = 4$ and $\omega = 4$



Notice how the solutions go to zero much more quickly here.

UNDER DAMPED: $p^2 < 4\omega$

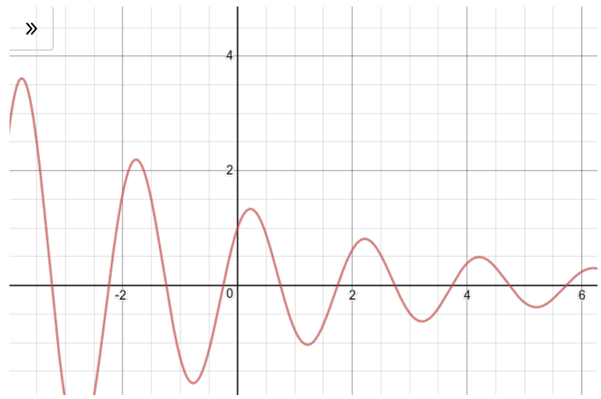
In this case our roots are complex. We have that $p^2 - 4\omega < 0$ so we will write roots as

$$r = \frac{-p}{2} \pm \frac{i}{2}\sqrt{-p^2 + 4\omega}$$

where I factored out the negative sign to get the i in front of the square root. Our solutions are

$$x(t) = e^{-\frac{pt}{2}}\left(c_1 \sin\left(\frac{t}{2}\sqrt{4\omega - p^2}\right) + c_2 \cos\left(\frac{t}{2}\sqrt{4\omega - p^2}\right)\right)$$

Now what do we expect for our solutions? Here we see the solutions are decaying thanks to the $e^{-pt/2}$ out front, but we also see oscillations. In this case your spring is stronger than the damping so oscillations happen but are slowed down over time. Think about a really old car when the shocks start going out. If you jump off the bumper it will bounce for a while but eventually slow down. When your shocks go out the go from being critically damped, damping out oscillations as quickly as possible, to being under damped and bouncing. Notice here that this is a very fine line! Here is an example graph for when $p = \frac{1}{2}$ and $\omega = 10$



As you can see the roots of the characteristic equation tell us everything about the behavior, or characteristics, of the physical system!

Case 2 - Forced Motion - Nonhomogeneous

Now we will consider what happens when there is some forcing in the system. For a real life example think back to the car from the homogeneous case. Now instead of jumping off and letting the system respond naturally you bounce on the bumper, keeping some forcing on the system throughout time. One good example is when people install hydraulics! They are supposed to "force" the car to keep bouncing!

For our discussion here we will keep it simple and talk about Undamped Forced Oscillations. This means that $p = 0$ giving us the equation

$$x'' + \omega x = f(x)$$

Here it is easy to solve the characteristic equation. In fact we have solved this one SO MANY TIMES! We find that $r = \pm\sqrt{-\omega}$. Here notice that we only have two cases $\omega = 0$ meaning that there is no spring or $\omega > 0$ meaning there is a real life spring. Let's look at the more interesting case of having a spring. In this case we would have imaginary roots and solutions to the homogeneous equation would be

$$x_c = c_1 \sin(\sqrt{\omega}t) + c_2 \cos(\sqrt{\omega}t)$$

Now because we have forcing we need to find the nonhomogeneous solution. Here there are two possible cases. For MUC we make a guess for x_p based on the form of $f(t)$ and either x_p does not duplicate our homogeneous solution and we are fine, OR x_p does duplicate the homogeneous solution and we need to multiply by t to get an linearly independent guess.

No Duplications

Here lets assume that $f(t) = F_0 \cos(\alpha t)$ where $\alpha \neq \sqrt{\omega}$ in this case we would guess

$$x_p = A \sin(\alpha t) + B \cos(\alpha t)$$

taking derivatives and plugging this into our ODE we find

$$A(1-\alpha^2)\sin(\alpha t) + B(1-\alpha^2)\cos(\alpha t) = F_0 \cos(\alpha t)$$

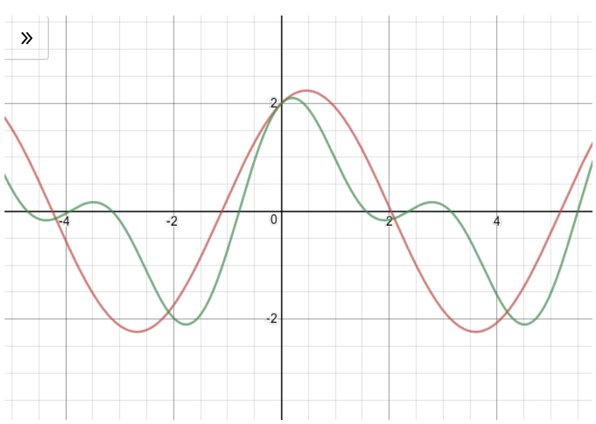
Solving gives

$$x_p = \frac{F_0}{1-\alpha^2}\cos(\alpha t)$$

So our solutions would be

$$x(t) = c_1 \sin(\sqrt{\omega}t) + c_2 \cos(\sqrt{\omega}t) + \frac{F_0}{1-\alpha^2}\cos(\alpha t)$$

Here the forcing is changing the oscillations making them either faster or slower. Here we can graph the case of $\omega = 1$, $\alpha = 2$ and let $F_0 = 0$ or $F_0 = 1$ to see the difference between the forced and unforced cases.



Here the red solution is the homogeneous unforced solution and the green solution is forced solution. Notice how the green solution oscillates differently but it's amplitude does not grow or decay.

Duplications

Now lets assume that $f(t) = F_0 \cos(\alpha t)$ where $\alpha = \sqrt{\omega}$ in this case we would guess

$$x_p = At \sin(\sqrt{\omega}t) + Bt \cos(\sqrt{\omega}t)$$

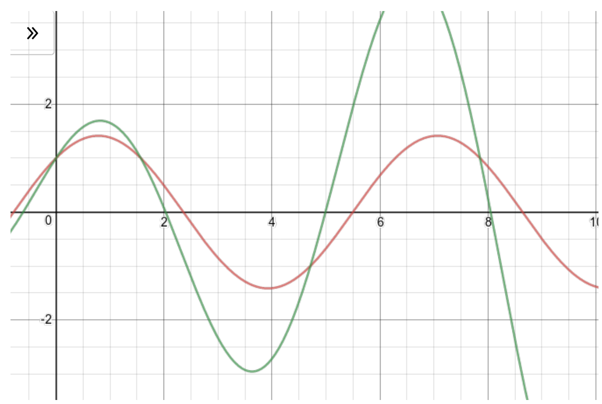
Notice how I had to multiply by t to remove the duplication! Here I can plug in my guess and solve for the coefficients to find

$$x_p = \frac{F_0}{2\sqrt{\omega}} t \sin(\sqrt{\omega}t)$$

well this changes everything! Now what is happening to our solution?

$$x(t) = c_1 \sin(\sqrt{\omega}t) + c_2 \cos(\sqrt{\omega}t) + \frac{F_0}{2\sqrt{\omega}} t \sin(\sqrt{\omega}t)$$

The nonhomogeneous term grows as time increases. Here is a graph for the case of $\omega = 1$, $\alpha = 1$ and let $F_0 = 0$ or $F_0 = 1$



This effect is called resonance. Think about when you were a kid swinging on a swing. If you just randomly swing your legs you lose speed pretty quickly. But, if you swing your legs at just the right frequency... what happens... your oscillations grow because you are forcing them at the resonant frequency!

This is all fun and games if you are on a swing, but the idea of resonant frequency can have serious implications. Think about what would happen if a buildings natural resonant frequency exactly matched the frequency of an earthquake!

One very famous example is the Tacoma Narrows bridge where the frequency of the gusting winds exactly matched the resonant frequency of the bridge. What do you think happened? Yeah, the bridge broke apart and fell down in a wind storm! Now given, this bridge is much more complicated than our simple one dimensional equation... but that's why you should take Partial Differential Equations!

The Moral(s) of the Story

- There are really interesting implications for what the nonhomogenous solution does in realy life physical systems.
- The characteristic equation really does tell you about the characteristics of the physical system!
- Resonance is an important physical phenomenon.