

Differential Equations – Notes

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Office Hours:

Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations – Day 19

So far we are able to solve a few classes of higher order ODEs.

- Linear Constant Coefficient (use the Characteristic Equation and MUD or VOP)
- Euler Equations (use $v = \ln(x)$ to transform and VOP)
- Reduction of Order if we are missing either the independent or dependent variable completely.

But what about other types of equations. For example how would I solve

$$xy'' + y' + xy = 0$$

Right now none of our solution method will work. This is why we will talk about integral transform methods.

Laplace Transform

The Laplace Transform is a special kind of integral transform. The basic idea behind the integral transform is that you "plug" your ODE into a special integral, do the integration, usually by parts, and as a result you have a much simpler problem to solve. In the case of the Laplace Transform, we are going to reduce our ODE to an algebraic problem!

First we will start by just introducing the Laplace Transform and we will work to get comfortable with the idea.

The Laplace Transform Given a function $f(t)$ defined for all $t \geq 0$ the Laplace Transform of f is the function F defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} (f(t)) dt$$

for values of s where the integral converges.

NOTE: Because this is an improper integral we must first do the integral and then consider convergence!

EXAMPLE:

Find the Laplace Transform of $f(t) = 1$ for $t > 0$.

$$\mathcal{L}[1] = \int_0^{\infty} e^{-st} (1) dt$$

Notice that taking the Laplace Transform literally means plug the thing you want to transform into the integral.

Now we need to integrate. Here is your chance to practice the improper integrals that you learned in Calculus II!

$$\begin{aligned} \lim_{b \rightarrow \infty} \left[\int_0^b e^{-st} dt \right] &= \lim_{b \rightarrow \infty} \left[\frac{-1}{s} e^{-st} \Big|_0^b \right] = \\ &= \lim_{b \rightarrow \infty} \left[\frac{-1}{s} e^{-sb} + \frac{1}{s} \right] = \frac{1}{s} \end{aligned}$$

as long as $s > 0$ notice that this integral does not converge if $s \leq 0$. We will almost always get restrictions on the value of s from evaluating the limit in the improper integral here. We are always seeking values of s for which the Laplace Integral converges.

EXAMPLE:

Find the Laplace Transform of $f(t) = e^{at}$ for $t \geq 0$.

$$\mathcal{L}[e^{at}] = \int_0^{\infty} e^{-st} (e^{at}) dt$$

$$\begin{aligned} \lim_{b \rightarrow \infty} \left[\int_0^b e^{(a-s)t} dt \right] &= \lim_{b \rightarrow \infty} \left[\frac{1}{a-s} e^{(a-s)t} \Big|_0^b \right] = \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{a-s} e^{(a-s)b} - \frac{1}{a-s} \right] = \frac{-1}{a-s} \end{aligned}$$

as long as $s > a$ we have that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

NOTE Before the transformation t is the independent variable and after the transformation s is the independent variable. This will be very important when solving ODEs!

YOU TRY:
Find the Laplace Transform of $f(t) = t$. ANSWER ¹

The Gamma Function

The Gamma Function is a fun and useful function to know. It can really help to simplify how we write our Laplace Transform answers and it's just plain cool.

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

Notice how much this looks like the Laplace Transform. This function shows up in many places in math, quite often in probability models. There are some rules for this function:

$$\begin{aligned}\Gamma(1) &= 1 \\ \Gamma(x+1) &= x\Gamma(x)\end{aligned}$$

What does this mean? Well let's think about

$$\Gamma(n+1) = n\Gamma(n)$$

but if we continue with this logic

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2)$$

so really we could write

$$\Gamma(n+1) = n(n-1)(n-2) \cdots (2)\Gamma(1) = n(n-1)(n-2) \cdots (2)(1)$$

If n is an integer, what function is this? That's right! The Factorial! So this means that $\Gamma(x)$ is a generalization of the Factorial function $n!$ and that we can calculate things like

$$1/3! = \Gamma(1/3)$$

Now let's use this in the Laplace Transform.

EXAMPLE:

Find the Laplace Transform of $f(t) = t^a$ where a is any real number such that $a > -1$ and $t \geq 0$.

$$\mathcal{L}[t^a] = \int_0^{\infty} e^{-st} (t^a) dt$$

Okay how do we solve this integral? We can actually use a substitution! Let $u = st$ then $t = \frac{u}{s}$ and $dt = \frac{1}{s} du$. When $t = 0$ we have $u = 0$ and as $t \rightarrow \infty$ we have $u \rightarrow \infty$. So we can change our integral to

$$\begin{aligned}\int_0^{\infty} \frac{u^a}{s^a} e^{-u} \frac{du}{s} &= \int_0^{\infty} \frac{u^a}{s^{a+1}} e^{-u} du = \\ \frac{1}{s^{a+1}} \int_0^{\infty} u^a e^{-u} du &= \frac{1}{s^{a+1}} \Gamma(a+1)\end{aligned}$$

Here because we recognized the Gamma Function, we didn't even need to do the integral. We can just write

$$^1F(s) = \frac{1}{s^{a+1}}, \quad s > 0$$

the transform with the Gamma Function!

This example actually just gave us a general formula for the Laplace Transform. We found that

$$\mathcal{L}[t^a] = \frac{1}{s^{a+1}} \Gamma(a+1), \quad a > -1$$

How can we use this? Well what would the Laplace Transform of $f(t) = t^2$ be?

$$\mathcal{L}[t^2] = \frac{1}{s^{2+1}} \Gamma(2+1) = \frac{1}{s^3} \Gamma(3)$$

And using what we know about the relationship between the Gamma Function and factorials we can write

$$\frac{1}{s^3} \Gamma(3) = \frac{3!}{s^3} = \frac{6}{s^3}$$

There are many **Tables of Laplace Transforms** that contain formulas like this one. We will hand one example out in class, but even more detailed tables can be found in books or online. The big issues that most students have is in reading and learning how to use the table transforms. So let's practice a few.

EXAMPLE:

Find the Laplace Transform of $f(t) = \sin(3t + 5)$

On the table we see #15

$$f(t) = \sin(at + b)$$

and to the right of it is written the transform formula

$$F(s) = \frac{s \sin(b) + a \cos(b)}{s^2 + a^2}$$

So what we really need to do is define $a = 3$ and $b = 5$ and then plug these into the formula.

$$F(s) = \mathcal{L}(\sin(3t + 5)) = \frac{s \sin(5) + 3 \cos(5)}{s^2 + 3^2}$$

Remember that s is our new independent variable which is why we often write $F(s) = \mathcal{L}(f(t))$.

As usual this can get more tricky as our functions get more complicated and we will always be working on ways to simplify our expressions. So please be careful and make sure you really understand now for the simple cases how the table is used.

Properties of the Laplace Transform

There are many useful properties for this integral transform. The first very useful property is that the Laplace Transform is Linear.

LINEARITY:

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t))$$

Basically all the normal rules of integration apply! You can bring constants out front and apply the transform separately to terms in a sum.

The next very useful property is that the Laplace Transform has an inverse. To us this means that we can transform and ODE, solve it in transform space, and then transform back using the inverse! More on this tomorrow. For now

INVERSE:

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s) ds$$

This inverse is defined by the Complex Valued Bromwich Integral. In practice we **USE A TABLE** to find these inverses. But this is a good chance for me to tell you to take Complex Analysis where we learn about complex numbers and integration.

EXAMPLE:

Find the inverse Laplace Transform of $F(s) = \frac{4}{s^2 + 4}$. For our class we will ALWAYS use a table to find the inverse transform. Looking at our table this matches with #7 so we see that if

$$F(s) = \frac{a}{s^2 + a^2}$$

then

$$f(t) = \mathcal{L}^{-1}(F(s)) = \sin(at)$$

so the inverse transform of our

$$F(s) = \frac{4}{s^2 + 4} = 2 \frac{2}{s^2 + 4}$$

is given by

$$f(t) = 2 \sin(2t)$$

Notice here how I had to rewrite $F(s)$ a little bit to match what I found on the table!

EXISTENCE and UNIQUENESS As long as $f(t)$ is piecewise continuous for $t \geq 0$ and is of exponential order as $t \rightarrow \infty$ then the Laplace Transform exists. The Laplace transform is unique, meaning that if $F(s) = \mathcal{L}(f(t))$ and $G(s) = \mathcal{L}(g(t))$ then if $F(s) = G(s)$, we are guaranteed $f(t) = g(t)$

PIECEWISE CONTINUOUS FUNCTIONS One beautiful thing about the Laplace Transform is its ability to transform piecewise functions. For us this becomes useful when our forcing term in an ODE is an impulse or turns on and off.

$${}^2F(s) = \frac{1}{s^2} [1 - se^{-s} - e^{-s}]$$

What do I mean by a Piecewise Continuous Function?

$f(t)$ is piecewise continuous on a bounded interval $a \leq t \leq b$ provided that it can be divided into finitely many subintervals such that

1. $f(t)$ is continuous on each of the subintervals
2. $f(t)$ has a finite limit as it approaches the end points of the subintervals

EXAMPLE:

The Unit Step Function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

What does this function look like? A step that goes from zero to one at $t = 0$.

EXAMPLE:

Find the Laplace Transform of the shifted unit step function

$$u(t - a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

The Laplace Transform is just an integral so when in doubt just do the integral.

$$\begin{aligned} \mathcal{L}(u(t - a)) &= \int_0^\infty e^{-st} (u(t - a)) dt \\ &= \int_0^a e^{-st} (0) dt + \int_a^\infty e^{-st} (1) dt \\ &= \lim_{b \rightarrow \infty} \left[\int_a^b e^{-st} (1) dt \right] = \lim_{b \rightarrow \infty} \left[\frac{-1}{s} e^{-sb} + \frac{1}{s} e^{-sa} \right] \\ &= \frac{1}{s} e^{-sa} \quad s \geq 0, a \geq 0 \end{aligned}$$

YOU TRY:

First plot and then find the Laplace Transform for the piecewise continuous function

$$f(t) = \begin{cases} t & t < 1 \\ 0 & t \geq 1 \end{cases}$$

ANSWER ²

The Moral(s) of the Story

- Laplace Transforms (and integral transforms in general) are hugely useful in solving differential equations and in many other applications.
- To take the Laplace Transform we either just **Do The Integral** or we look it up on a table.
- To find the Inverse Laplace Transform, we look it up on a table.

