# POWER SERIES.



Solving ODE ... methods so far

Lots of first order methods: Just integrate, Seperable, Exact
Integrating factor, Reduction / Subst.

Higher Order: Reduction of Order, Characteristic + MUC/VOP,
Variable transformation, Laplace Transform.

What if all of those fail? Power Series or Numerical Methods.

Bessels Eqn of order $n$ $\quad x^2y'' + xy' + (x^2 - n^2)y = 0$

acoustics, heat flow, electromagnetic radiation.

Legendre's Eqn. $\quad (1-x^2)y'' - 2xy' + n(n-1)y = 0$

Power series methods give solutions to these odes in the form of an infinite series.

REVIEW OF POWER SERIES $\quad \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a)^n + c_2(x-a)^2 + \cdots$

power series centered at $x=a$
often concerned with case $a=0$.

$f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges on the interval $I$ provided the limit exists.

$$\lim_{N\to\infty} \sum_{n=0}^{N} c_n x^n$$

many elementary functions have power series representations. these are called Taylor Series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cdots$$

Ex $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ geometric series.

is the taylor series for $\frac{1}{1-x}$.

Does this always converge?

$\lim_{N\to\infty} \sum_{n=0}^{N} x^n$

only exists when $|x|<1$

Some Operations...

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

then $f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$ easy!

$$f(x)\,g(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \cdots$

more complicated.
"Infinite Foil"

THE POWER SERIES METHOD.

given an ODE... assume the solution has the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

sub this in and then determine $c_n$'s that satisfy the ODE.

not always successful.

we will need $y' = \sum_{n=0}^{\infty} c_n n x^{n-1}$

if $y$ converges on some open interval $I$ then $y$ is differentiable on $I$.

NOTE if $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$

for every point $x \in I$ then $a_n = b_n$ for all $n$.

LETS TRY A SIMPLE EXAMPLE!

Ex $y' + 4y = 0$

We can already solve this one!

$$\frac{dy}{dx} = -4y$$

$$\frac{dy}{y} = -4dx$$

$$\ln|y| = -4x + c$$

$$y = Ae^{-4x}.$$

let's use this to see how power series work!

SUB IN $y$ and $y'$

$$\sum_{n=1}^{\infty} c_n n x^{n-1} + 4\sum_{n=0}^{\infty} c_n x^n = 0$$

we really want to factor out $x^n$

SHIFT INDEX to get $x^n$

let $n \to n+1$ let $n \to n$

$$\sum_{n=0}^{\infty} c_{n+1}(n+1)x^n + 4\sum_{n=0}^{\infty} c_n x^n = 0$$

Shift Index.

a closer look....

$$\sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 x^0 + c_2 2x^1 + c_3 3x^2 + \dots = \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n$$

write out series

new pattern starting w/ $n=0$. and $x^n$

COMBINE THE SERIES.

$$\sum_{n=0}^{\infty} \left( c_{n+1}(n+1) + 4c_n \right) x^n = 0$$

The coefficients make this TRUE

$$c_{n+1}(n+1) + 4c_n = 0$$

so $$c_{n+1} = -\frac{4c_n}{(n+1)}$$

WE GET A RECURRANCE RELATION FOR $c_n$'s

let $n=0$ then.

$$c_1 = -\frac{4c_0}{1}$$

$n=1$ $$c_2 = -\frac{4c_1}{2} = \frac{-4(-4c_0)}{2\cdot 1} = \frac{4^2 c_0}{2\cdot 1}$$

LOOK FOR PATTERN!

$n=2$ $$c_3 = -\frac{4c_2}{3} = \frac{-4(4^2 c_0)}{3\cdot 2\cdot 1} = \frac{-4^3 c_0}{3\cdot 2\cdot 1}$$

$n=3$ $$c_4 = \frac{-4c_3}{4} = \frac{-4(-4^3 c_0)}{4\cdot 3\cdot 2\cdot 1} = \frac{4^4 c_0}{4\cdot 3\cdot 2\cdot 1}$$

We see that $C_n = \frac{(-1)^n (4)^n C_0}{n!} = \frac{(-1)^n 4^n C_0}{n!}$

$$= \frac{(-4)^n C_0}{n!}$$

plug this into our $y$...

$$y = \sum_{n=0}^{\infty} \frac{(-4)^n C_0}{n!} x^n = \sum_{n=0}^{\infty} \frac{C_0 (-4x)^n}{n!}$$

$$= C_0 \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} = C_0 e^{-4x}$$

It WORKS! YAY!

You will want to get good at this "SHIFT OF INDEX" idea!

$$\sum_{n=3}^{\infty} a_n (n-1) x^{n-2}$$

Shift to $x^n$ need $(n+2)-2$ to get there.
so $n \to n+2$ EVERY WHERE.

$$\sum_{n=1}^{\infty} a_{n+2} (n+1) x^n$$

$n+2=3$, $n=1$ $\quad$ $n+2-1$, $n+1$ $\quad$ $n+2-2$, $n$

Check BOTH REPRESENTATIONS give the same thing!

$$a_3 (2) x^1 + a_4 (3) x^2 + \cdots = a_3 (2) x^1 + a_4 (3) x^2 + \cdots$$

YOU TRY... $(x-3)y' + 2y = 0$ $\quad$ assume $y = \sum_{n=0}^{\infty} c_n x^n$

then $y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$

$$(x-3) \sum_{n=1}^{\infty} c_n n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} c_n n x^n - 3 \sum_{n=1}^{\infty} c_n n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n = 0$$

SHIFT

$$\sum_{n=0}^{\infty} c_n n x^n - 3 \sum_{n=0}^{\infty} c_{n+1} (n+1) x^n + 2 \sum_{n=0} c_n x^n = 0$$

doesn't effect the sum!

so $\displaystyle\sum_{n=0}^{\infty} \left[ C_n n - 3C_{n+1}(n+1) + 2C_n \right] x^n = 0$

which leaves $\quad C_n n - 3C_{n+1}(n+1) + 2C_n = 0$

$$-3C_{n+1}(n+1) = -2C_n - nC_n$$

$$C_{n+1} = \frac{2C_n + nC_n}{3(n+1)}$$

$$C_{n+1} = \frac{(n+2)\,C_n}{3(n+1)}$$

look for a pattern...

$n=0 \qquad C_1 = \dfrac{2C_0}{3\cdot 1}$

$n=1 \qquad C_2 = \dfrac{3C_1}{3(2)} = \dfrac{3\cdot 2\ C_0}{3\cdot 2\cdot 3\cdot 1} = \dfrac{3.2\,C_0}{3^2\ 2.1}$

$n=2 \qquad C_3 = \dfrac{4\,C_2}{3\cdot 3} = \dfrac{4\cdot 3\cdot 2\cdot C_0}{3^2\cdot 2\cdot 1\cdot 3\cdot 3} = \dfrac{4\cdot 3\cdot 2\ \ C_0}{3^3\ 3.2.1}$

$$C_n = \frac{(n+1)!\ C_0}{3^n\ n!} = \frac{(n+1)\,C_0}{3^n}$$

soln: $\quad y = \displaystyle\sum_{n=0}^{\infty} C_0\,\frac{(n+1)}{3^n}\,x^n = C_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n}\,x^n$

Check convergence ...

RADIUS OF CONVERGENCE

given $\quad y = \displaystyle\sum_{n=0}^{\infty} C_n x^n \qquad\qquad \rho = \lim_{n\to\infty} \left| \frac{C_n}{C_{n+1}} \right|$ radius of convergence.

IF $\rho = 0$ the series diverges

IF $0 < \rho < \infty$ the series converges for $|x| < \rho$ and diverges for $|x| > \rho$

IF $\rho = \infty$ the series converges for all $x$

For the problem above

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{n+1}{3^n}}{\frac{n+1+1}{3^{n+1}}} \right| = \left| \frac{n+1}{3^n} \cdot \frac{3^{n+1}}{n+2} \right| = \left| \frac{3n+3}{n+2} \right| = 3$$

So the series converges for $-3 < x < 3$.

Ex Solve $x^2 y' = y - x - 1$

with a power series...

$$x^2 \sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = -x-1$$

$$\sum_{n=1}^{\infty} c_n n x^{n+1} - \sum_{n=0}^{\infty} c_n x^n = -x-1$$

Separate out unique terms.

so starts at $x^2$ starts at $c_0 x^0$ needs to balance $-x$ and $-1$

$$\sum_{n=1}^{\infty} c_n n x^{n+1} - c_0 - c_1 x - \sum_{n=2}^{\infty} c_n x^n = -x-1$$

Gather $x^n$ SHIFT IF NEEDED.

SHIFT so $n=2$ $n-1=1$ $n=2$.

$n \to n-1$

GATHER TERMS.

$$\sum_{n=2}^{\infty} c_{n-1}(n-1)x^n - \sum_{n=2}^{\infty} c_n x^n - c_0 - c_1 x = -x-1$$

$$\sum_{n=2}^{\infty} \left[c_{n-1}(n-1) - c_n\right] x^n - c_0 - c_1 x = -x-1$$

so $c_0 = 1$, $c_1 = 1$, and $(n-1)c_{n-1} - c_n = 0$

SOLVE. or $c_n = (n-1)c_{n-1}$ $n \geq 2$

$n=2$ $c_2 = 1 \cdot c_1 = 1 \cdot 1$

$n=3$ $c_3 = 2 \cdot c_2 = 2 \cdot 1 \cdot 1$

$n=4$ $c_4 = 3 \cdot c_3 = 3 \cdot 2 \cdot 1 \cdot 1$

so $c_n = (n-1)!$ $n \geq 2$

so our soln

$$y(x) = 1 + x + \sum_{n=2}^{\infty} (n-1)!\, x^n$$

CHECK CONVERGENCE:

$$\rho = \lim_{n\to\infty} \left| \frac{(n-1)!}{(n+1-1)!} \right| = \lim_{n\to\infty} \left| \frac{(n-1)!}{n!} \right| = \lim_{n\to\infty} \left| \frac{1}{n} \right| = 0$$

only converges if $|x| = 0$

THIS ODE DOES NOT HAVE A CONVERGENT POWER SERIES SOLN !!!

our assumption $y = \sum_{n=0}^{\infty} c_n x^n$ was false in this case.