Differential Equations

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 2

Today we begin learning about solution methods. Often for a given problem there is more than one way to solve it. Remember, we are trying to predict the future of the universe so be creative in your solution approaches!

Integrals as Solutions

When our ODE has the form

$$\frac{dy}{dx} = f(x)$$

in other words we have a first order equation that is a function of ONLY the dependent variable, then the simplest solution approach would be to just integrate. Hey, I'm so glad I practiced all those integrals yesterday!

Remember! For almost all of the problems in this class, once you have a solution you can always plug it back into the original equation to test our answers.

For each problem we look to

1. Find the general solution - this will contain one or more unknown constants.
2. Then if initial conditions are given we look for the particular solution - these conditions allow us to solve for the unknown constants.

EXAMPLE:

$$\frac{dy}{dx} = (4x^2+1)^2 x, \quad y(0) = 1$$

Okay I will do this in steps to show you what goes through my head:

1. Classify the equation. What kind of beast are we dealing with? Well this is first order, linear, and I see only x's on the RHS. So I could probably just integrate!
2. Apply the solution method:

$$\int \frac{dy}{dx}\, dx = \int (4x^2+1)^2 x\, dx$$

$$\text{let } u = 4x^2+1 \text{ then } du = 8x dx$$

$$y(x) = \frac{1}{8}\int u^2\, du = \frac{1}{24}u^3 + c \tag{1}$$

This give us a general solution of

$$y(x) = \frac{1}{24}(4x^2+1)^3 + c$$

3. Now we apply the initial condition to find the particular solution. We know that $y(0) = 1$ so this means that when $x = 0$, $y = 1$. We just need to plug these two things into our general solution!

$$1 = \frac{1}{24}(4(0)^2+1)^2 + c$$

so solving for c we find $c = 1 - \frac{1}{24} = \frac{23}{24}$ and our particular solution is

$$y(x) = \frac{1}{24}(4x^2+1)^3 + \frac{23}{24}$$

4. Now check your solution! This must satisfy $\frac{dy}{dx} = (4x^2+1)^2 x$ so if we take the first derivative of our solution we should get the RHS of our ODE. Does it work?

YOU TRY:

$$\frac{dy}{dx} = 2x+3 \quad y(1) = 2$$

Don't forget to check your answer [1]

[1] You should get $y(x) = x^2 + 3x - 2$

General vs Particular Solutions

Here is an example to show how important initial conditions are in real world problems.

EXAMPLE:

Imagine that some drops a water balloon from a dorm room window, that just happens to be right over your head. It accelerates toward you under the influence of gravity.

From physics we know $\vec{F} = m\vec{a}$ where $\vec{F}$ is the force, in this case gravity, m is the mass of the object, and $\vec{a}$ is the acceleration. We are going to assume no air resistance, because I am the boss! This means that for the force we can write $\vec{F} = -mg\hat{j}$ where the $\hat{j}$ just implies that the force in working only in the vertical direction, AKA they dropped it straight down. We also all should know that acceleration is the second derivative of position $\frac{d^2y}{dt^2}$ in the vertical direction so this gives us an ODE

$$-mg = -m\frac{d^2y}{dt^2}$$

or

$$\frac{d^2y}{dt^2} = g$$

where g is a constant. We can solve this using our "just integrate it" method from above!

Because we have a second derivative we would integrate twice:

$$\frac{dy}{dt} = gt + c_1$$
$$y(t) = \frac{1}{2}gt^2 + c_1t + c_2$$

Okay great, how fast do you need to get out of the way? Why is this an impossible question to answer?

What we have is a general solution, which is great because we can walk all over the world and apply it to all similar water balloon capers, we don't need to re-solve the ODE. BUT, each time we need to give some other info. What info do we need? In this case we have two unknown constants so we need two pieces of information. The initial position, or what floor the balloon was thrown from and the initial velocity, of how hard did they chuck it at us.

Let's use the initial conditions

$$y(0) = 20, \quad y'(0) = 1$$

and assume our units are x in meters and t in seconds. What do these initial conditions mean? Let's plug in to find our particular solution and get out of the way of this falling watter balloon already!

$$y(0) = 20 \to 20 = \frac{1}{2}g(0)^2 + c_1(0) + c_2 = c_2 \to c_2 = 20$$
$$y'(0) = 1 \to 1 = -g(0) + c_1 \to c_1 = 1$$

For the second condition I took the derivative of the general solution before I plugged in. So our particular solution is

$$y(t) = -gt^2 + t + 20$$

It is easy to forget your initial conditions. Always double check whether or not initial conditions were given.

Slope Fields (AKA Direction Fields) and Solution Curves

Now let's explore GENERAL and PARTICULAR solutions from a graphical approach. This will help us visualize our solutions and it is a good way to get an idea of what to expect from your solution, AKA check answers.

The following will work for any ODE of the form

$$\frac{dy}{dt} = f(x, y)$$

here we just need a first order equation, but it can be any function of the independent or dependent variables.

First some questions:

1. In terms of the graph of a function what does $\frac{dy}{dx}$ represent?
2. So if I asked you to draw a function that matched $\frac{dy}{dx} = 1$ what would you draw? Draw an example of this function. Did everyone draw exactly the same thing?
3. Given answers to the above, how could we "draw" $\frac{dy}{dx} = x$

YOU TRY: Seriously, answer the questions. I even left space at the bottom of this page!

SLOPE FIELDS are a graphical way of thinking about the solutions to ODEs. What our first order differential equation is telling us is the slope of the function everywhere. Let's use this to our advantage and sketch out the slope at many places in the xy-plane.

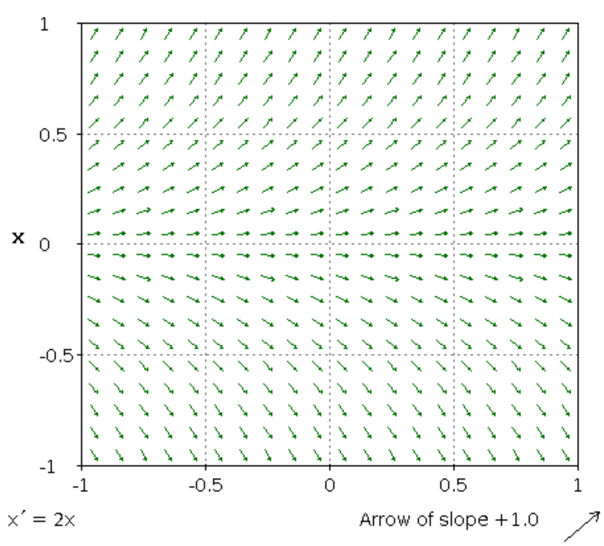
EXAMPLE:

$$\frac{dy}{dx} = 2y$$

Start with a table of values:

x	y	$\frac{dy}{dx}$
-	0	0
-	1	2
-	2	4
-	3	6
-	4	8
-	5	0
-	6	0

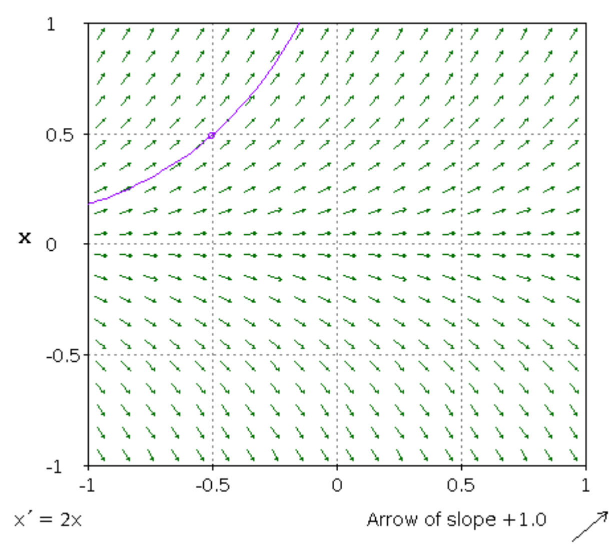
Here we don't need to input x values because our ODE doesn't depend on x, in other words along one line in x all the slopes are the same. Now let's draw the xy-plane and sketch in some of these slopes:



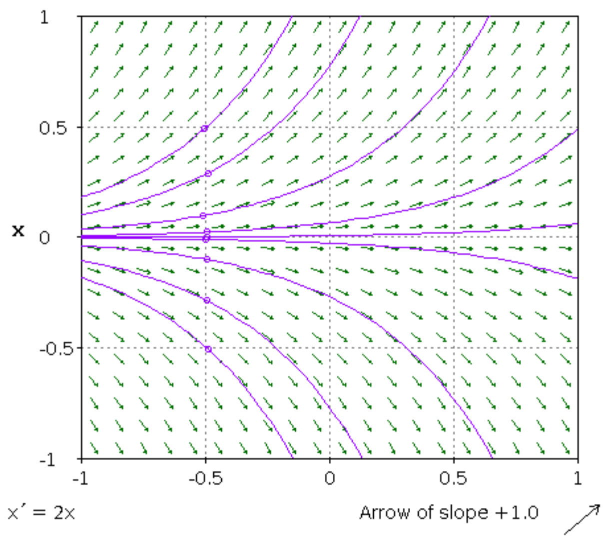
Okay so I cheated and used a computer program (I'll do this by hand in class), but how do we draw this? Here are some steps to get you started:

- Draw the xy-axis like we always do for graphing. Remember to label them!
- Choose some xy-value. How about $x = 0,\ y = 0$? Put your pencil there on the graph.
- Now glance at the table we made what should the slope be when $y = 0$? It should be zero, so we can draw a little arrow with slope zero here.
- Now pick another xy-value. How about $x = 1\ y = 0$? Put your pencil there on the graph. What should the slope be? Zero! We can draw a little arrow with slope zero.
- Notice a pattern? All of the little arrows along the x-axis will have slope zero! Draw in a bunch of these!
- Now pick another xy-value. How about $x = 0\ \ y = 1$? Put your pencil there on the graph.
- Look at the table what should the slope be? It should be two so we can draw a little arrow with slope two here.
- Just keep going like this until you have filled in the space reasonably well.

How do we use this weird graph? Well these little slope arrows act a guides for where our solution would be. Imagine that the arrows are pointing out a current for a river and you dropped a toy boat in the river. Where would it go? Lets see. Choose a point, we'll start with $(0.5, -0.5)$, put your pencil there on the graph and then start sketching a curve letting the slope and arrows guide your line. You can go both forward and backward from your starting point. You should get something like this:



What you just did was select a particular solution out of the sea of all possible solutions! Where you put your pencil initially is your initial condition. In our example this is $y(-0.5) = 0.5$. Technically you could pick all sorts of initial conditions and graph a representative set of possible solution curves:



As you can imagine this can get very time consuming. I only draw slope fields by hand when I find myself in a math argument at a restaurant and I'm trying to prove a point on a back of a napkin! Lucky for us there is a fabulous program called **Dfield**. You can find a download at:

http://math.rice.edu/~dfield/dfpp.html

Or by googling "dfield". Download the .jar file and run the program. I will show you how to do this in class. For your homework you are allowed to plot the slope fields using this program, either copy them by hand or print them out to add to your assignment.
YOU TRY:

$$y' = x - y$$

Draw a slope field by hand first just to check that you know how. Then compare what you drew to what you get from the program **dfield.jar**

The Moral(s) of the Story

- In the simplest case you can just integrate to find the solution to an ODE. The key is that you can write the equation such that you have the derivative on the LHS and only the independent variable is showing on the RHS.
- We talked about this in terms of first order equations but we could do simple higher order equations this way. For example,

$$\frac{d^7y}{dx^7} = 3x$$

  could be solved this way. We would just need to integrate SEVEN times and we would have SEVEN unknown constants!
- For any FIRST ORDER equation we can plot a SLOPE or DIRECTION field. We just recognize that the first derivative tells us the slope and we draw the slope arrows everywhere in the xy-plane.
- There are two types of solutions that we can find GENERAL and PARTICULAR. The general solution has unknown constants meaning that we can apply it to a lot of similar problems that have different starting conditions. The particular solution has initial conditions specified so all of the constants have been solved and it only apples to that particular starting point.