Differential Equations - Notes

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 20
Today we learn how to transform an ODE using the Laplace Transform.

ODEs and that Laplace Transform

Here is the general idea of what we will be doing

1. We start with a general ode
$$x'' + p(t)x' + q(t)x = f(t)$$
2. We take the Laplace Transform of the WHOLE ODE
$$\mathcal{L}[x''] + \mathcal{L}[p(t)x'] + \mathcal{L}[q(t)x] = \mathcal{L}[f(t)]$$
3. This will result in a simpler equation in terms of s. So we solve for
$$\mathcal{L}[x] = F(s)$$
4. the solution to our ODE is given by
$$x = \mathcal{L}^{-1}[F(s)]$$

The big question we are going to run into is: how do we take the Laplace Transform of the Derivative?

Laplace Transform of the Derivative

Let's consider
$$\mathcal{L}[x'] = \int_0^\infty e^{-st}\left(x'\right)\ dt$$
and think about how we might take this derivative. Well we could reduce that derivative if we did integration by parts! Here we will let $u = e^{-st}$ and $dv = x'dx$ because when we integrate dv the x' will go away. This gives $du = -se^{-st}$ and $v = x$.

$$\begin{aligned}\int_0^\infty e^{-st}\left(x'\right)\ dt &= \lim_{b\to\infty}\left[x(t)e^{-st}\Big|_0^b + s\int_0^b e^{-st}x(t)\ dt\right]\\ &= 0 - x(0) + s\int_0^\infty e^{-st}\left(x(t)\right)\ dt\\ &= x(0) + s\mathcal{L}[x]\end{aligned}$$

But this is good news! Because $x(0)$ is just our initial condition and we have transformed away our derivative being left with only the Laplace Transform of x and some other algebraic terms.

NOTE: Here we assumed that whatever our solution is $\lim_{b\to\infty} x(b)$ does not grow faster than e^{-st} decays. Otherwise our improper integral in the Laplace Transform above will not converge. We assume that we CAN do the Laplace Transform of $x(t)$.

YOU TRY:
Now it's your turn to try this method on a slightly harder problem, the second derivative! Please find
$$\mathcal{L}[x'']$$
Before you start, how many times will we need to do integration by parts to reduce the second derivative? That's right! TWICE.
ANSWER [1]

We can actually derive a general formula for the Laplace Transform of any derivative. If you want a challenge you could prove this formula!

$$\begin{aligned}\mathcal{L}[f^{(n)}] = s^n\mathcal{L}[f] - s^{n-1}f(0) - s^{n-2}f'(0) - \\ \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)\end{aligned}$$

Notice here that an n^{th} order ODE will have exactly $n-1$ initial conditions so this formula is consistent with what we expect from our ODEs.

[1] $\mathcal{L}[x''] = \int_0^\infty e^{-st}\left(x''\right)\ dt = s^2\mathcal{L}[x] - x'(0) - sx(0)$

Solving ODEs with the Laplace Transform

Let's jump right in and consider an example.

EXAMPLE:
Solve the initial value problem (IVP) using a Laplace Transform

$$x'' - x' - 6x = 0, \;\; x(0) = 2 \;\; x'(0) = 1$$

We always start by transforming THE WHOLE ODE. That means transforming each piece since we know that the Laplace Transform is linear.

$$\begin{aligned}
\mathcal{L}[x''] = s^2\mathcal{L}[x] - sx(0) - x'(0) = s^2\mathcal{L}[x] - 2s - 1 \\
\mathcal{L}[x'] = s\mathcal{L}[x] - x(0) = s\mathcal{L}[x] - 2 \\
\mathcal{L}[6x] = 6\mathcal{L}[x] \\
\mathcal{L}[0] = 0
\end{aligned}$$

Plugging all of this into our ODE we find

$$s^2\mathcal{L}[x] - 2s - 1 - s\mathcal{L}[x] + 2 - 6\mathcal{L}[x] = 0$$

Often we write the Laplace transform of a function as just the capital letter to make this notation nicer. This makes our problem

$$s^2X - 2s - 1 - sX + 2 - 6X = 0$$

This is a algebraic equation for our unknown function $X(s)$ we can simply solve this.

$$\begin{aligned}
(s^2 - s - 6)X + (2s - 1) = 0 \\
X = \frac{2s - 1}{s^2 - s - 6}
\end{aligned}$$

Now comes often the hardest part of the problem, figuring out the inverse transform. Our solution is just

$$x(t) = \mathcal{L}^{-1}[X(s)]$$

but when we look at our table we don't see anything that matches exactly what we have here! We actually need to use algebra to rewrite these function. Here I will use partial fractions to get

$$X(s) = \frac{3/5}{s-3} + \frac{7/5}{s+2}$$

I know to do this anytime I see something that is easy to factor in the denominator of my $F(s)$. Now I can transform back each piece using the table transform #2

$$f(t) = e^{at} \qquad F(s) = \frac{1}{s-a}$$

For us the fractions in the numerator just go out front and we need to decide what a should be

$$a = 3 \qquad \mathcal{L}^{-1}\left[\frac{3/5}{s-3}\right] = \frac{3}{5}e^{3t}$$
$$a = -2 \qquad \mathcal{L}^{-1}\left[\frac{7/5}{s+2}\right] = \frac{7}{5}e^{-2t}$$

Our solution is given by the sum of these

$$x(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}$$

One cool thing to notice here is that our initial conditions are already applied! This is our full particular solution to the ODE.

YOU TRY:

Here is a slightly harder example that you should try

$$x'' + 4x = sin(3t), \quad x(0) = x'(0) = 0$$

Here are some midway check points:

1. The Laplace Transform of the ode should give

$$s^2X - x'(0) - sx(0) = 4X = \frac{3}{s^2+9}$$

   where I used the formula for the derivatives and a table for the $\sin(3t)$

2. Solving for X we should get

$$X(s) = \frac{3}{(s^2-4)(s^2+9)}$$

   These two things in the denominator remind me A LOT of the sin and cos transforms. So we should do partial fractions to separate them!

3. Our solution should be

$$x(t) = \frac{3}{10}\sin(2t) - \frac{1}{5}\sin(3t)$$

There is not a typo in that solution, our cosine term is actually set to zero by the initial conditions, but it all happened behind the curtains of the Laplace Transform.

Tricks of the Laplace Transform

As you can see in these problems, the hardest part is either doing the transform, or more likely undoing the transform. There are many tricks for doing these transforms. Both today and next class we will work on using some of these ideas.

EXAMPLE:

Exploiting the derivative rule. If you are transforming a function where $f(0) = 0$ then you can exploit the transform of the derivative to make your job easier. Here we will find the transform of

$$f(t) = te^{at}$$

First, I know this one is on the fancy table, but it is a good example to work on! We need to check that $f(0) = 0$, which is does. That means that our trick will work. So here goes!

We don't really want to plug $f(t) = te^{at}$ in the the Laplace Transform integral, it would be a pain to integrate. So instead we consider the derivative rule.

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

and since $f(0) = 0$ we see

$$\mathcal{L}[f'(0)] = s\mathcal{L}[f(t)]$$

this means that if it is easier to transform the derivative of $f(t)$ then we should transform that instead or that we can exploit this relationship! Let's look at our function

$$f(t) = te^{at} \qquad f'(t) = e^{at} + ate^{at}$$

so using the relationship

$$\mathcal{L}[e^{at} + ate^{at}] = s\mathcal{L}[te^{at}]$$

But how does this help us? Well, let's do some simplification using the fact that the Laplace Transform is linear

$$\mathcal{L}[e^{at}] + a\mathcal{L}[te^{at}] = s\mathcal{L}[te^{at}]$$

and we can take the transform of e^{at} easily, we did this in class yesterday!

$$\frac{1}{s-a} + a\mathcal{L}[te^{at}] = s\mathcal{L}[te^{at}]$$

Now, solving for $\mathcal{L}[te^{at}]$ we find

$$\mathcal{L}[te^{at}] = \frac{\frac{1}{s-a}}{s-a} = \frac{1}{(s-a)^2}$$

Often times you will have simple transforms memorized and it is much easier to use fun tricks like this to find the transform!

Derivatives of the Laplace Transform

We can actually take the derivatives inside the Laplace Transform! Here is the formula for this

$$\mathcal{L}[-tf(t)] = F'(s)$$

where $F'(s)$ means that we found the Laplace Transform of $f(t)$ and then took the derivative with respect to s. How did we get this? Well we just took a derivative!

$$F(s) = \int_0^\infty e^{-st}\,(f(t))\;dt$$

We can take the derivative with respect to s

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}\,(f(t))\;dt$$

but because the integral is with respect to t we can pull the derivative inside and take derivatives!

$$F'(s) = \int_0^\infty \frac{d}{ds}e^{-st}f(t)\;dt = \int_0^\infty (-t)e^{-st}f(t)\;dt$$

the derivative just gives us an extra $-t$ each time so we find that

$$\mathcal{L}[-tf(t)] = F'(s)$$

We could easily generalize this for higher order derivatives

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

This idea is important in solving equations like

$$tx'' + x' + tx = 0, \quad x(0) = 1 \;\; x'(0) = 0$$

Notice that this is our FIRST non-constant coefficient problem that we will solve without reducing to a constant coefficient problem! This is a VERY SPECIAL equation it is Bessel's Equation of order zero. Let's solve it with a Laplace Transform.

We start by taking the transform of each piece and we use the derivative rule above. For the first term the derivative of a transform rule gives

$$\mathcal{L}[tx''] = -\frac{d}{ds}\mathcal{L}[x''] = -F'(s)$$

We can do the Laplace transform of the second derivative

$$\mathcal{L}[x''] = s^2\mathcal{L}[x] - sx(0) - x'(0) = s^2\mathcal{L}[x] - s$$

So combining the two ideas we find that

$$\mathcal{L}[tx''] = -\frac{d}{ds}(s^2\mathcal{L}[x] - s) = -2s\mathcal{L}[x] - s^2\frac{d}{ds}\mathcal{L}[x] + 1$$

here I used the product rule to take that derivative. Simplifying I find

$$\mathcal{L}[tx''] = -2sX - s^2X' + 1$$

The second term is straightforward, just like before

$$\mathcal{L}[x'] = s\mathcal{L}[x] - x(0) = s\mathcal{L}[x] - 1 = sX - 1$$

The third term again needs the derivative rule

$$\mathcal{L}[tx] = -\frac{d}{ds}\mathcal{L}[x] = -X'$$

Now we put all of this together to transform the ODE

$$\begin{aligned} -2sX - s^2X' + 1 - sX - 1 - X' &= 0 \\ -(s^2+1)X' - sX &= 0 \\ (s^2+1)X' + sX &= 0 \end{aligned}$$

Because we started with a more complicated ODE we actually were not able to reduce down to an algebraic equation! Instead we reduced to a lower order ODE, which we can solve! This equation is first order and separable:

$$\frac{dX}{ds} = \frac{-s}{s^2+1}X$$
$$\frac{dX}{X} = \frac{-s}{s^2+1}ds$$

The solution to this ODE is given by

$$X(s) = \frac{C}{\sqrt{s^2+1}}$$

okay so we have two problems. First we have an unknown constant, so that means we are going to need to figure out this constant by applying the initial conditions again after we transform back. Second we need to transform this thing! This is not an easy transform so here is an outline of what we would do (NOTE: You are not expected to be able to do something like this on an exam!)

We start by rewriting $X(s)$

$$X(s) = \frac{C}{s}\left(1+\frac{1}{s^2}\right)^{-1/2}$$

Then we would expand this thing in the binomial series

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots$$

This would give us an infinite series that we can take the inverse transform of piece by piece, but if we enforce the condition that $x(0) = 0$ then we find that $C = 1$ in this series. So we get our solution

$$x(t) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$$

It is a bit of a mess, but it gives us a solution to our ODE.

For us, in the homework on on an exam, we can leave our solution in the form

$$x(t) = \mathcal{L}^{-1}\left[\frac{C}{\sqrt{s^2+1}}\right]$$

So basically if you CAN find the inverse then do so and write the solution nicely. If you cannot find the inverse then just write the solution as represented above.

The Moral(s) of the Story

- We can use the Laplace Transform to transform ODEs into much simpler algebraic equations!
- Often the hardest part of these problems is finding the inverse transform at the end. My best advise is be patient and do partial fraction decomposition!
- The place that people make the most mistakes is in the algebra. GO SLOW!