Continue Power Series ...

Example from Last Time: $x^2 y' = y - x - 1$ see previous notes.

How does this work for SECOND ORDER?

Ex $\quad y'' + y = 0 \qquad$ we know the soln $y = c_1 \sin(x) + c_2 \cos(x)$!

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

sub in $\quad \displaystyle\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$

SHIFT $n \to n+2$

so $x^{n-2} \to x^n$

$$\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

BOTH START AT $n=0$ ✓

$$\sum_{n=0}^{\infty} \left[c_{n+2}(n+2)(n+1) + c_n\right] x^n = 0$$

then $\quad c_{n+2}(n+2)(n+1) + c_n = 0$

or $\quad c_{n+2} = \dfrac{-c_n}{(n+2)(n+1)} \qquad n \geq 0$

$n=0$ then $\quad c_2 = -\dfrac{c_0}{2\cdot 1} \qquad\qquad n=1$ then $\quad c_3 = -\dfrac{c_1}{3\cdot 2}$

$n=2 \quad c_4 = -\dfrac{c_2}{4\cdot 3} = \dfrac{+c_0}{4\cdot 3\cdot 2\cdot 1} \qquad\qquad n=3 \quad c_5 = \dfrac{-c_3}{5\cdot 4} = \dfrac{+c_1}{5\cdot 4\cdot 3\cdot 2}$

$n=$ even $\Rightarrow 2n \qquad$ TWO UNKNOWN CONSTANTS c_0 AND c_1 $\qquad n=$ odd $\Rightarrow 2n+1$

$$c_{2n} = \frac{(-1)^n c_0}{2n!} \qquad\qquad c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!}$$

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n c_0}{2n!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n c_1}{(2n+1)!} x^{2n+1}$$

$$= c_0 \cos(x) + c_1 \sin(x) \ldots$$

IDEA WE WILL USE LATER...

Imagine we had never heard of the functions sine and cosine...

Then $y = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

would be the best we could do.

<u>BUT</u> we can actually understand a lot about these mystery functions from their infinite sums!

say we call $C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$ and $S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n-1)!}$

COULD WE APPLY INITIAL CONDITIONS, $x=0$? $\rightarrow$ SURE

$$C(0) = \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n}}{2n!} = 1$$

when $n=0$ the leading term x^0 survives.

$$S(0) = \sum_{n=0}^{\infty} \frac{(-1)^n (0)^{2n+1}}{(2n+1)!} = 0$$

what about convergence? SURE

$C(x)$

$$\lim_{n\to\infty} \left[\rho = \left| \frac{\frac{1}{2n!}}{\frac{1}{(2n+2)!}} \right| = \left| \frac{2(n+1)!}{2n!} \right| \right] = \lim_{n\to\infty} \left|(2n+2)(2n+1)\right| = \infty$$

$\rho = \infty \Rightarrow C(x)$ converges for all x

$S(x)$

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(2n+1)!}}{\frac{1}{(2n+2+1)!}} \right| = \lim_{n\to\infty} \left| \frac{(2n+3)!}{(2n+1)!} \right| = \lim_{n\to\infty} \left|(2n+3)(2n+2)\right| = \infty$$

$\rho = \infty \Rightarrow S(x)$ converges for all x.

What about Derivatives... derivatives should exist when/where the series converges.

$$C'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{2n!} = -\sum_{n=1} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = -S(x)$$

derivative of cosine $= -$ sine !

We could also show $S'(x) = C(x)$.

So we could say the the functions $C(x)$ and $S(x)$ are fully determined by the differential eqn

$$y'' + y = 0$$

that they are linearly independent solns to !

In this case this feels silly because we are totally used to SINE AND COSINE as functions.

But there are many functions that are defined this way:

BESSELS FUNCTIONS, LEGENDRE POLYNOMIALS, etc.

Now so far we have mostly seen the "Good News" cases.

IN GENERAL we want to solve:

$$A(x)y'' + B(x)y' + C(x)y = 0$$

we can always use VOP for y_{NH}.

-OR-

$$y'' + P(x)y' + Q(x)y = 0$$

where $P = \frac{B}{A}$ $Q = \frac{C}{A}$

If $A(x) \to 0$ for some values of x then $P(x)$ and $Q(x)$ fail to be analytic.

<u>Ex</u> $xy'' + y' + xy = 0$ -or- $y'' + \frac{1}{x}y' + y = 0$

We can see that there will be problems near $x=0$

$P(x)$ is not analytic.

$x = a \neq 0$ is called an <u>ordinary point</u>

$x = 0$ is called a <u>singular point</u>

We have to be very careful solving ODES near singular points!

$y = \sum_{n=0}^{\infty} c_n x^n$ would fail

$y = \sum_{n=0}^{\infty} c_n (x-a)^n$

Radius of convergence is at least as large as distance between a and nearest singular point.

CHOOSE a from K's. $a \neq 0$ would likely succeed.

Ex $(x^2-4)y'' + 3xy' + y = 0$
$y(0) = 4$
$y'(0) = 1$

ASSUME
$$y = \sum_{n=0}^{\infty} C_n (x-0)^n = \sum_{n=0}^{\infty} C_n x^n$$

IC applied at $x_0 = 0$

CHECK FOR SINGULAR POINTS !

$\frac{3x}{x^2-4}$ and $\frac{1}{x^2-4}$ both singular when $x = \pm 2$

BUT we are solving near $x=0$ so we should have a radius of convergence of at least 2.

NOW do power series method.

$$(x^2-4)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + 3x\sum_{n=1}^{\infty} C_n n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n = 0$$

DISTRIBUTE

$$\sum_{n=2}^{\infty} C_n n(n-1)x^n - \sum_{n=2}^{\infty} 4C_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} 3C_n n x^n + \sum_{n=0}^{\infty} C_n x^n = 0$$

SHIFT $n \to n+2$

$$\sum_{n=2}^{\infty} C_n n(n-1)x^n - \sum_{n=0}^{\infty} 4C_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 3C_n n x^n + \sum_{n=0}^{\infty} C_n x^n = 0$$

IF $n=0$ I get
$0 + 0 + C_2(2)(1)x^2 + \cdots$
ok to start $n=0$

IF $n=0$ I get
$0 + 3C_1(1)x + \cdots$
ok to start at $n=0$.

WRITE AS MUCH AS POSSIBLE UNDER ONE SUM w/ x^n Factored out.

$$\sum_{n=0}^{\infty} \left[C_n n(n-1) - 4C_{n+2}(n+2)(n+1) + 3C_n n + C_n\right]x^n = 0$$

$$-4C_{n+2}(n+2)(n+1) + \left[n(n-1) + 3n + 1\right]C_n = 0$$

or
$$C_{n+2} = \frac{[n^2+2n+1]C_n}{4(n+2)(n+1)} = \frac{(n+1)^2 C_n}{4(n+2)(n+1)}$$

so
$$C_{n+2} = \frac{(n+1)C_n}{4(n+2)} \qquad n \geq 0$$

Look for a pattern.

$n=0 \quad C_2 = \dfrac{C_0}{4 \cdot 2}$ $\qquad$ $n=1 \quad C_3 = \dfrac{2C_1}{4 \cdot 3}$

$n=2 \quad C_4 = \dfrac{3C_2}{4 \cdot 4} = \dfrac{3C_0}{4 \cdot 4 \cdot 4 \cdot 2} = \dfrac{3C_0}{4^2 \cdot 4 \cdot 2}$ $\qquad$ $n=3 \quad C_5 = \dfrac{4C_3}{4 \cdot 5} = \dfrac{4 \cdot 2 C_1}{4 \cdot 4 \cdot 5 \cdot 3} = \dfrac{4 \cdot 2 C_1}{4^2 5 \cdot 3}$

$n=4 \quad C_6 = \dfrac{5C_4}{4 \cdot 6} = \dfrac{5 \cdot 3 C_0}{4^3 \cdot 6 \cdot 4 \cdot 2}$ $\qquad$ $n=5 \quad C_7 = \dfrac{6C_5}{4 \cdot (7)} = \dfrac{6 \cdot 4 \cdot 2 C_1}{4^3 \cdot 7 \cdot 5 \cdot 3}$

$n=1,2,3\ldots$ $\quad$ $n=0$ we get $C_0 x^0$ $\qquad$ $n=1,2,3\ldots$ $\quad$ $n=0$ we get $C_1 x^1$

so $\quad C_{2n} = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)\, C_0}{4^n \cdot [2 \cdot 4 \cdot 6 \cdots (2n)]}$

EVEN CASES

so $\quad C_{2n+1} = \dfrac{2 \cdot 4 \cdot 6 \cdots (2n+1-1)\, C_1}{4^n \; 3 \cdot 5 \cdot 7 \cdots (2n+1)}$

$= \dfrac{2 \cdot 4 \cdot 6 \cdots (2n)\, C_1}{4^n \; 3 \cdot 5 \cdot 7 \cdots (2n+1)}$

$= \dfrac{2^n (n)!\, C_1}{4^n \, (2n+1)!!} = \dfrac{2^n n!\, C_1}{4^n (2n+1)!!}$

we write $\quad 1 \cdot 3 \cdot 5 \cdots (2n-1) = (2n-1)!!$

just short hand.

and $\quad 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n \cdot n!$

$2(1 \cdot 2 \cdot 3 \cdots n)$

so $\quad C_{2n} = \dfrac{(2n-1)!!\, C_0}{4^n \cdot 2 \cdot n!}$

$$\text{so} \quad y(x) = C_0\left(1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!\, C_0}{2^n 4^n n!} x^{2n}\right) + C_1\left(x + \sum_{n=1}^{\infty} \frac{2^n n!\, C_1\, x^{2n+1}}{4^n (2n+1)!!}\right)$$

Apply IC's...

$y(0) = C_0 = 4$ $\qquad$ $y' = C_0 \sum_{n=1}^{\infty} \dfrac{(2n-1)!!\, C_0}{2^n 4^n n!} 2n x^{2n-1} + C_1 + \sum \cdots = C_1 = 1$

so

$$y(x) = 4 + 4\sum_{n=1}^{\infty} \frac{(2n-1)!!\, C_0}{2^n 4^n n!} x^{2n} + x + 2\sum_{n=1}^{\infty} \frac{n!\, C_1}{2^n 4^n (2n+1)!!} x^{2n+1}$$

NEXT LEGENDRE EQUATION.