Differential Equations - Notes

Professor:
Dr. Joanna Bieri
joanna_bieri@redlands.edu

Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 21
More practice with the Laplace Transform.

As terms in our transform start to get more complicated we need to get good at noticing patterns to help us transform back. For example it is good to memorize the basic transform pairs.

$$f(t) = e^{kt} \qquad F(s) = \frac{1}{s-k}$$

$$f(t) = 1 \qquad F(s) = \frac{1}{s}$$

$$f(t) = \sin(kt) \qquad F(s) = \frac{k}{s^2+k^2}$$

$$f(t) = \cos(kt) \qquad F(s) = \frac{s}{s^2+k^2}$$

for the $\sin(t)$ and $\cos(t)$ notice that they always have a plus in the denominator, if you see a minus then you are probably dealing with $\sinh(t)$ or $\cosh(t)$

$$f(t) = t^n \qquad F(s) = \frac{n!}{s^{n+1}}$$

Translations on the s-axis

Often when trying to find an inverse transform we are really close to one of the forms above except instead of s we have $(s-a)$. For example transforming

$$F(S) = 1/(s-3)^5$$

This looks a lot like the transform of $f(t) = t^4$ but we have that darn -3 in the denominator. Luckily there is a rule for this.

When your transform is in the form $F(s-a)$, this represents a translation on the s-axis. In t-space this represents multiplying by e^{at}. Here is the rule

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

in other words we find the transform of $f(t)$ then plug in $s = s-a$. Going the other direction

$$e^{at}f(t) = \mathcal{L}^{-1}[F(s-a)]$$

Why does this work? Let's use the integral definition to see this property at work

$$\mathcal{L}[e^{at}f(t) = \int_0^\infty e^{-st}\left(e^{at}f(t)\right)\, dt = \int_0^\infty e^{-(s-a)t}f(t)dt$$

but this is just the Laplace transform if we let $a = s-a$ or translate along the s-axis. In other words

$$\int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a)$$

Thanks to this property we can transform things like

$$f(t) = e^{at}\sin(kt) \qquad F(s) = \frac{k}{(s-a)^2+k^2}$$

$$f(t) = e^{at}\cos(kt) \qquad F(s) = \frac{(s-a)}{(s-a)^2+k^2}$$

$$f(t) = e^{at}t^n \qquad F(s) = \frac{n!}{(s-a)^{n+1}}$$

EXAMPLE:
Find the inverse Laplace Transform for

$$F(s) = \frac{1}{(s-2)^2+4}$$

First we notice that this looks a lot like the $\sin(kt)$ case, we are just off by a constant and have the shift in the denominator. So we can let $k=2$ and $a=2$ to find the inverse transform

$$f(t) = \frac{1}{2}e^{2t}\sin(2t)$$

Where the $1/2$ came from the fact that $F(s)$ only had a 1 in the numerator and the transform needed k in the numerator.

NOTE: Often these shifts in $\sin(kt)$ and $\cos(kt)$ will come after you complete the square in the denominator of $F(s)$!

Tricks with Partial Fractions

At this point in the class most people are thinking "If I have to do another Partial Fractions problem I am going to egg Joanna's house!" We'll because I don't like to waste food here is a nice trick for Partial Fractions.

Rather than multiplying by the denominator and getting n equations for n unknowns. We will instead exploit the fact that we know the roots of the denominator to simplify our partial fractions. **Beware! If you have repeated roots, then this method breaks down a bit and you need to use derivatives!**

EXAMPLE:
Find the inverse transform of

$$F(s) = \frac{s^2+1}{s^3+2s^2-8s}$$

We know to do partial fractions because the order of the denominator is greater than the order of the numerator. So, factoring the denominator we find $s(s+4)(s-2)$. The roots to this polynomial are $s=0$, $s=-4$ and $s=2$. Our guess for the form of the solution is

$$\frac{s^2+1}{s(s+4)(s-2)} = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s-2)}$$

now we multiply by the denominator to get

$$s^2+1 = A(s+4)(s-2) + Bs(s-2) + Cs(s+4)$$

Next we will plug each of the roots into this equation.

$s=0$

$$1 = A(4)(-2) \qquad A = \frac{-1}{8}$$

notice how plugging in the root got rid of all the other terms!

$s=-4$

$$16+1 = B(-4)(-6) \qquad B = \frac{-17}{24}$$

$s=2$

$$4+1 = C(6)(2) \qquad C = \frac{5}{12}$$

This leaves us with

$$F(s) = \frac{-1/8}{s} - \frac{17/24}{s+4} + \frac{5/12}{s-2}$$

and we can transform this simply using the table or our memory!

$$f(t) = -\frac{1}{8} - \frac{17}{24}e^{-4t} + \frac{5}{12}e^{2t}$$

If you had a repeated root you would need to use the derivative to find those constants. We will show this in the next example!

EXAMPLE:
Solve the following ODE using the Laplace Transform

$$y'' + 4y' + 4y = t^2 \qquad y(0)=0 \quad y'(0)=0$$

First we need to transform each piece

$$\mathcal{L}[y''] = s^2Y$$
$$\mathcal{L}[4y'] = 4sY$$
$$\mathcal{L}[4y] = 4Y$$
$$\mathcal{L}[t^2] = \frac{2}{s^3}$$

Putting this all back together in the ODE we find

$$(s^2+4s+4)Y = \frac{2}{s^3}$$

and solving for Y we get

$$Y = \frac{2}{s^3(s^2+4s+4)}$$

To transform back we are going to need to do partial fractions! Factoring the denominator we get $s^3(s+2)^2$ which gives a triple root at $s=0$ and a double root at $s=-2$. Let's see how our trick works in this case. We start with our guess

$$\frac{2}{s^3(s+2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{(s+2)} + \frac{E}{(s+2)^2}$$

We will start doing this the "normal" way...

Multiplying through by the denominator gives

$$2 = As^2(s+2)^2 + Bs(s+2)^2 + C(s+2)^2 + Ds^3(s+2) + Es^3$$

Now we will do our trick of plugging in the roots.

$s=0$
Plugging this in gives

$$2 = C(2)^2$$

So we find that $C = \frac{1}{2}$.

But we missed out on TWO OTHER SOLUTIONS, since $s=0$ is a triple root. We actually still need to find two more constants from this root!

The normal way breaks down and we need to go about this a little differently.

The trick with multiple roots has to do with taking derivatives! Going back to our original equation

$$\frac{2}{s^3(s+2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{(s+2)} + \frac{E}{(s+2)^2}$$

We will first consider the $s = 0$ root:

We start by multiplying just by s^3 since we are looking at the $s = 0$ root. Doing this we find

$$\frac{2}{(s+2)^2} = As^2 + Bs + C + s^3 g(s)$$

I am using a short cut here and realizing that all those messy terms in $g(s)$ will be set to zero when I plug in $s = 0$. Let's try again plugging in $s = 0$

$$\frac{2}{4} = C$$

and again we get $C = \frac{1}{2}$. Now take a derivative of our equation with respect to s to get

$$\frac{-4}{(s+2)^3} = 2As + B + 2s^2 g(s) + s^3 g'(s)$$

plugging in $s = 0$ gives

$$\frac{-4}{8} = B$$

and now we find $B = \frac{-1}{2}$. Notice how taking derivatives here is revealing those extra constants. Because we have a triple root we need to take one more derivative.

$$\frac{12}{(s+2)^4} = 2A + sh(s)$$

I am using $sh(s)$ as a place holder for the derivative of $2s^2 g(s) + s^3 g'(s)$ whatever this is I know it will be multiplied by at least one s so there is no need for me to do the extra work! Plugging in $s = 0$ we get

$$\frac{12}{16} = 2A$$

and we find $A = \frac{3}{8}$. Now we have all three constants associated with the triple root $s = 0$.

YOU TRY:
What happens when we apply this method to the other root

$$r = -2$$

How many derivatives will we need to take for a double root?
Here you should multiply by $(s+2)^2$ and find

$$\frac{2}{s^2} = D(s+2) + E + (s+2)^2 f(s)$$

were I am using f(s) to hold all the stuff multiplied by $(s+2)^2$ that is going to go to zero every time.

Using the method and plugging in $s = -2$ I should find $D = \frac{-3}{8}$ and $E = \frac{-1}{4}$.

After finishing our partial fractions we find that

$$Y = \frac{3/8}{s} + \frac{-1/2}{s^2} + \frac{1/2}{s^3} + \frac{-3/8}{(s+2)} + \frac{-1/4}{(s+2)^2}$$

Now we need to look for patterns and figure out what the inverse transform is. First I see terms like $\frac{1}{s^n}$ so I know these will give me things like t^{n-1} the other terms I see are either just a standard $\frac{1}{s-a}$ but on of them is raised to a power so really it is shifted by $a = -2$. This gives me the solution to my ode as

$$y(t) = \frac{1}{4}t^2 - \frac{1}{2}t + \frac{3}{8} - \frac{1}{4}te^{-2t} - \frac{3}{8}e^{-2t}$$

YOU TRY:

Solve the ODE

$$x'' + 6x' + 34x = 0$$

using the Laplace Transform. HINT: you will need to complete the square before transforming back!

Halfway Hints:

$$X(s) = \frac{3s+19}{s^2+6s+34} = \frac{3s+19}{(s+3)^2+25}$$

this can be broken up into two pieces, a shifted $\sin(5t)$ and a shifted $\cos(5t)$.

ANSWER: [1]

The Moral(s) of the Story

- The hardest part of these problems is often the algebra after taking the transform. The more you practice the better you will get.
- If you have some of the transforms memorized it is easier to do the algebra because you will know what form you would like your solution to be in. **GOAL ORIENTED ALGEBRA!**
- Go slow and think about one part of the problem at a time.

[1] $x(t) = e^{-3t}(3\cos(5t) + 2\sin(5t))$