Legendre Polynomials

~~[illegible]~~

Legendre Eqn. $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

where $\alpha > -1$

Are there any singular points?

$$P(x) = \frac{-2x}{(1-x^2)} \quad \text{and} \quad Q(x) = \frac{\alpha(\alpha+1)}{(1-x^2)}$$

so $x = \pm 1$ is a singular point.

So this ode should have two linearly independent solutions in the form of the powerseries sentered at $a=0$

$$y = \sum_{n=0}^{\infty} C_n x^n$$

with a radius of convergence of at least one.

PLUG IN $(1-x^2)\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} C_n n x^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} C_n x^n = 0$

$$\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} C_n n(n-1)x^n - \sum_{n=1}^{\infty} 2C_n n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} C_n x^n = 0$$

Shift
need $n-2 \to n$
let $n \to n+2$

$$\sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} C_n n(n-1)x^n - \sum_{n=1}^{\infty} 2C_n n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} C_n x^n = 0$$

PEAL OFF $n=0$ and $n=1$ TERMS...

$n=0$: $2C_2 + \alpha(\alpha+1)C_0 = 0$ so $C_2 = -\dfrac{\alpha(\alpha+1)C_0}{2}$

$n=1$: $[6C_3 - 2C_1 + \alpha(\alpha+1)]x = 0$ so $C_3 = -\dfrac{\alpha(\alpha+1)+2}{3\cdot 2}C_1$

$n=2^+$ $[C_{n+2}(n+2)(n+1) - C_n n(n-1) - 2C_n n + \alpha(\alpha+1)C_n]x^n = 0$

$$C_{n+2} = -\frac{\alpha(\alpha+1) + n(n-1) + 2n}{(n+2)(n+1)}C_n = -\frac{(\alpha-n)(\alpha+n+1)}{(n+2)(n+1)}C_n \quad n \geq 2$$

works for $n=0$, $n=1$!

$n=0 \quad C_2 = -\frac{\alpha(\alpha+1)C_0}{2!}$ $\qquad$ $n=1 \quad C_3 = -\frac{(\alpha-1)(\alpha+2)}{3!}$

$n=2 \quad C_4 = -\frac{(\alpha-2)(\alpha+3)}{4\cdot 3}C_2$ $\qquad$ $n=3 \quad C_5 = -\frac{(\alpha-3)(\alpha+4)}{(5)(4)}C_3$

$= +\frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}C_0$ $\qquad$ $= \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}C_1$

SWITCH FROM n to m.... for historical reasons... n = integer vals of α...

$C_{2m} = (-1)^m a_{2m} C_0$

where $a_{2m} = \frac{\alpha(\alpha-2)(\alpha-4)\cdots(\alpha-2m+2)\cdot(\alpha+1)(\alpha+3)\cdots\cdots(\alpha+2m-1)}{(2m)!}$

$C_{2m+1} = (-1)^m a_{2m+1} C_1$

where

$a_{2m+1} = \frac{(\alpha-1)(\alpha-3)\cdots\cdots(\alpha-2m+1)(\alpha+2)(\alpha+4)\cdots\cdots(\alpha+2m)}{(2m+1)!}$

so our two linearly independent solns are.

$y_1 = C_0 \sum_{m=0}^{\infty}(-1)^m a_{2m} x^{2m}$ and $y_2 = C_1 \sum_{m=0}^{\infty}(-1)^m a_{2m+1} x^{2m+1}$

BUT WHAT ARE THESE SPECIAL FUNCTIONS ?

Consider the case of $\alpha = n$ = positive integer and lets explore...

Lets write out a few terms.

$$y_1 = c_0\left[1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 - \cdots\right]$$

$$y_2 = c_1\left[x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!} - \cdots\right]$$

What if $\alpha = \text{even} = 0, 2, 4, \ldots$

$\alpha = 0$ then $y_1 = c_0$ degree 0 $\quad y_2 =$ infinite series

$\alpha = 2$ then $y_1 = c_0 - c_0\frac{2(2+1)}{2!}x^2 \quad y_2 =$ infinite series.

ONE OF THE SOLNS IS ALWAYS POLYNOMIAL EXACTLY.

What if $\alpha = \text{odd} = 1, 3, 5 \ldots$

$\alpha = 1$ then $y_1 =$ infinite series $\quad y_2 = c_1 x$

same thing except odd powers!

These POLYNOMIAL SOLNS are very special... LEGENDRE POLYNOMIALS!

often you see

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

where we sub $\alpha = n$.

FOR BOUNDARY VALUE PROBLEMS WE NEED OSCILLATING SOLNS... this is powerful!

$$P_n(x) = \sum_{k=0}^{N} \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

where $N = [\![n/2]\!]$ integer part of $n/2$

$P_0 \equiv 1 \qquad P_1 = x \quad N=0$

$P_2(x) = \frac{1}{2}(3x^2-1) \;\; N=1 \qquad P_3(x) = \frac{1}{2}(5x^3-3x) \;\; N=1$

SHOW SOME GRAPHS.