Frobenius Method

What do we have so far...

given $\quad y'' + P(x)y' + Q(x)y = 0$

$$y(a) = c_1$$
$$y'(a) = c_2$$

we can expand our solns as power series

$$y = \sum_{n=0}^{\infty} C_n (x-a)^n$$

which as a radius of convergence of at least the nearest singular point.

Singular points are values of x where $p(x)$ or $q(x)$ become undefined.

OK but ....

We have solved equations like:

$$x^2y'' + 2xy' + 3y = 0 \quad \text{Equidimensional}$$
$$y(0) = 1$$
$$y'(0) = 1$$

and we get "nice" solns of the form $y = x^r$ where r is a number real or complex.

Isn't $x=0$ a singular point?!? How can this work?

There are different kinds of singular points!

consider a singular point at $x=0$ $\quad \cancel{x^2y'' + xp(x)y' + x^2Q(x)y = 0}$

$\cancel{y'' + \frac{P(x)}{x} +}$

$$y'' + p(x)y' + Q(x)y = 0$$

we say that $P(x)$ has a weak singularity at $x=0$
if $P(x) \to \infty$ no more rapidly than $1/x$
and $Q(x) \to \infty$ no more rapidly than $1/x^2$

Say it find: Singularity at $x=0$ then I know $y=\sum_{n=0}^{\infty} c_n x^n$ won't work.

BUT I now check

$xP(x)$ as $x \to 0$ and $x^2 Q(x)$ as $x \to 0$

If these are analytic (not ∞) then

$x=0$ is a REGULAR SINGULAR POINT good NEWS !! :)

otherwise

$x=0$ is an IRREGULAR SINGULAR POINT bad NEWS !! :(

Ex Bessels Eqn (order zero)

$$x^2 y'' + xy' + x^2 y = 0$$

divide by x^2 $\quad y'' + \frac{1}{x} y' + y = 0$

so $P(x) = \frac{1}{x} \to \infty$ as $x \to 0$ $\quad x=0$ is a SINGULAR POINT

Check $xP(x) = 1 \to 1$ as $x \to 0$ it is a REGULAR SINGULAR POINT.

Ex $2x^3 y'' + (1+x)y' + 3xy = 0$

$P(x) = \frac{1+x}{2x^3} \to \infty$ as $x \to 0$

$Q(x) = \frac{3}{2x^2} \to \infty$ as $x \to 0$

Check $xP(x) = \frac{1+x}{2x^2} \to \infty$ as $x \to 0$ This means $x=0$ is an Irregular Singular Point

$x^2 Q(x) = 3 \to 3$ as $x \to 0$ BUT

Ex Equidimensional: $x^2 y'' + 2xy' + 3y = 0$

Check $x=0$!

Here solns were x^r but r does not need to be an integer x^n !

Let's allow some flexibility in our Power Series

let $y = x^r \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} C_n x^{n+r}$ FROBENIUS SERIES

now we need to find BOTH C_n and r!

BACK TO OUR GENERAL EQN

$$y'' + P(x)y' + Q(x)y = 0$$

This just lets us investigate in general $x=0$ as a regular singular point.

lets let $P(x) = \frac{x\,p(x)}{x^2}$ so that $x=0$ is a regular singular point

and $p(x) = p_0 + p_1 x + p_2 x^2 + \cdots$

and $Q(x) = \frac{q(x)}{x^2}$ again

and $q(x) = q_0 + q_1 x + q_2 x^2 + \cdots$

p and q are analytic everywhere so they have a powerseries!

so $x^2 y'' + x\,p(x)\,y' + q(x) = 0$ is our general form.

sub this all in!! (deep breath!)

$$y = \sum_{n=0}^{\infty} C_n x^{n+r} \quad \text{then} \quad y' = \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$$

$$\text{and} \quad y'' = \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2}$$

$$x^2 \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2} + (p_0 + p_1 x + p_2 x^2 + \cdots)\, x \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$$
$$+ (q_0 + q_1 x + q_2 x^2 + \cdots) \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r} + (p_0 + p_1 x + \cdots) \sum_{n=0}^{\infty} C_n (n+r) x^{n+r} + (q_0 + q_1 x + \cdots) \sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

smallest power of x happens when $n=0$ or at x^r...

then we will have x^{r+1}, x^{r+2}, etc...

coefficients for each must ballance $= 0$

$n=0$ terms:

$$C_0 r(r-1)x^r + p_0 C_0 r x^r + C_0 q_0 x^r = 0$$

so $$C_0\left[r(r-1) + p_0 r + q_0\right]x^r = 0$$

don't want $C_0 = 0$ so $r(r-1) + p_0 r + q_0 = 0$

"Indicinal Eqn" — like a Characteristic Eqn!

This allows us to solve for r-values.

HOW MANY: second order: r_1 and r_2

We are garaunteed at least one soln of the form

$$y = x^r \sum_{n=0}^{\infty} C_n x^n$$

corresponding to the LARGEST ROOT r_1 or r_2

$$y = \sum_{n=0}^{\infty} C_n x^{n+r_1}$$

CASES:

1. If $r_1 - r_2 \neq 0$ and not an integer

   then second soln is $y = \sum_{n=0}^{\infty} C_n x^{n+r_2}$

2. If $r_1 = r_2$ Then there is only ONE frobenius soln

   $$y = \sum_{n=0}^{\infty} C_n x^{n+r}$$

   an we generate another soln using more advanced methods

3. If $r_1 \geq r_2$ and $r_1 - r_2 =$ integer

   There might be another frobeneus soln corresponding to r_2

We will focus on CASE 1! The exceptional cases are much more Complicated 8.4.

Let's Do an example or Two...

First ... we often solve for r_1 and r_2 BEFORE plugging into the ODE w/ the series...

p_0 is the leading constant in the P polynomial
q_0 is the leading constant in the q polynomial.

Ex $\quad 2x^2(1+x)y'' + 3x(1+x)^3y' - (1-x^2)y = 0$

$$P(x) = \frac{3(1+x)^2}{2x} \quad \text{and} \quad Q(x) = \frac{-(1-x^2)}{2x^2(1+x)} = -\frac{(1-x)(1+x)}{2x^2(1+x)} = \frac{-(1-x)}{2x^2}$$

we see that $x=0$ is a singular point!

BUT since $P(x) \to \infty$ like $1/x$ and $Q(x) \to \infty$ like $1/x^2$

It is a <u>regular</u> singular point... METHOD OF FROBENIUS!

looking at these

$$P(x) = \frac{\frac{3}{2}(1+2x+x^2)}{x} \quad \text{and} \quad Q(x) = \frac{-\frac{1}{2}(1-x)}{x^2}$$

so $p_0 = 3/2$ and $q_0 = -1/2$ plug into our general indicial eqn.

$$r(r-1) + \tfrac{3}{2}r - \tfrac{1}{2} = 0$$

$$r^2 - r + \tfrac{3}{2}r - \tfrac{1}{2} = r^2 + \tfrac{1}{2}r - \tfrac{1}{2} = 0 \qquad (r+1)(r-\tfrac{1}{2}) = 0$$

so $r_1 = 1/2$ and $r_2 = -1$ since $r_1 - r_2 = 3/2$ not an integer or zero

so we are gauranteed TWO frobenius solns:

$$y = x^{1/2}\sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y = x^{-1}\sum_{n=0}^{\infty} c_n x^n$$

we can now sub in $y = x^r \sum_{n=0}^{\infty} c_n x^n$

but then use $r = 1/2$ and $r = -1$ so our algebra becomes easier.

Next Time Bessels Eqn order zero!