Full Frobenius Soln:

Ex $2x^2y'' + 3xy' - (x^2+1)y = 0$

$P(x) = \frac{3}{2x}$ and $Q(x) = -\frac{(1+x^2)}{2x^2}$ $x=0$ is a regular singular point

so $p(x) = \frac{3}{2}$ and $q(x) = -\frac{1}{2}(1+x^2)$

$p_0 = 3/2$ $q_0 = -1/2$

Find r_1 and r_2

$r(r-1) + p_0 r + q_0 = r^2 - r + \frac{3}{2}r - \frac{1}{2}$ $r_1 = 1/2$

$= r^2 + \frac{1}{2} - \frac{1}{2} = (r+1)(r-1/2) = 0$ $r_2 = -1$

since $r_1 - r_2 \neq 0$ or an integer, we will have two Frobenius solns.

$$y_1 = x^{1/2}\sum_{n=0}^{\infty} c_n x^n \quad \text{and} \quad y_2 = x^{-1}\sum_{n=0}^{\infty} c_n x^n$$

now lets find the constants c_n...

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \quad \text{general form.}$$

then $y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$ $y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$

so $2x^2\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r+2} + 3x\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} - (x^2+1)\sum_{n=0}^{\infty} c_n x^{n+r} = 0$

$$\sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} 3c_n(n+r)x^{n+r} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r+2}}_{\substack{\text{shift} \\ n \to n-2}} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$-\sum_{n=2} c_{n-2}x^{n+r}$$

OK so we have two special cases to peel off

$n=0$ and $n=1$ then we will have general $n \geq 0$

$n=0$ $\quad 2c_0 r(r-1) + 3c_0 r - c_0 = 0$

so $[2r(r-1)+3r-1]c_0 = 0$ or $2(r^2+\frac{1}{2}r-\frac{1}{2})c_0 = 0$

plug in r_1 and r_2 .. they make the first term go to zero so

c_0 survives without being set to zero

$n=1$ $\quad 2c_1(1+r)(r) + 3c_1(1+r) - c_1 = 0$

so $[2r(r+1)+3(r+1)-1]c_1 = 0$ or $(2r^2+5r+2)c_1 = 0$

r_1 and r_2 DO NOT make this go to zero!

$c_1 = 0$

general case:

$$2c_n(n+r)(n+r-1) + 3c_n(n+r) - c_{n-2} - c_n = 0$$

so $$c_n = \frac{c_{n-2}}{2(n+r)(n+r-1)+3(n+r)-1}$$

simplifying is worth your time!

You decide where to stop.

$$= \frac{c_{n-2}}{2(n^2+2nr+r^2-n-r)+3(n+r)-1}$$

$$= \frac{c_{n-2}}{2(n^2-2nr+r^2)-2(n+r)+3(n+r)-1}$$

$$c_n = \frac{c_{n-2}}{2(n+r)^2+(n+r)-1} \qquad n \geq 2$$

now we consider the two cases $r_1 = 1/2$ and $r_2 = -1$

CASE 1 $\quad r_1 = 1/2$ then $a_n = \dfrac{a_{n-2}}{2(n+1/2)^2+(n+1/2)-1} = \dfrac{a_{n-2}}{2(n^2+n+\frac{1}{4})+(n+\frac{1}{2})-1}$

$$= \frac{a_{n-2}}{2n^2+3n}$$

$n=2$ $\quad a_2 = \dfrac{a_0}{2(2)^2+3\cdot 2} = \dfrac{a_0}{14}$ $\qquad n=3.\ a_3 = \dfrac{a_1}{2(3)^2+3\cdot 3} = 0$

because $c_1 = a_1 = 0$

so $a_5 = a_7 = \dots = 0$

we would find $a_0 = a_0$ $a_2 = \frac{a_0}{14}$ $a_4 = \frac{a_0}{616}$ $a_6 = \frac{a_0}{55440}$

$n=0$ $n=2$ $n=4$ $n=6$

so the first FROBENEUS SOLN is

$$y_1(x) = a_0 x^{1/2}\left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \frac{x^6}{55440} + \cdots\right)$$

CASE 2 $r_2 = -1$

$$b_n = \frac{b_{n-2}}{2(n-1)^2 + (n-1) - 1} = \frac{b_{n-2}}{2n^2 - 4n + \cancel{2} + n - \cancel{1} - 1}$$

$$= \frac{b_{n-2}}{2n^2 - 3n}$$

$n=2$ $b_2 = \frac{b_0}{2(2)^2 - 3(2)} = \frac{b_0}{2}$ $n=3$ $b_3 = \frac{b_1}{\text{stuff}} = 0$

because $b_1 = 0$

so $b_0 = b_0$ $b_2 = \frac{b_0}{2}$ $b_4 = \frac{b_0}{40}$ $b_6 = \frac{b_0}{2160}$

the SECOND FROBENIUS SOLN IS

$$y_2(x) = b_0 x^{-1}\left(1 + \frac{x^2}{2} + \frac{x^4}{40} + \frac{x^6}{2160} + \cdots\right)$$

Bessels Eqn of ORDER ZERO.

– we solved this using Laplace Transforms

Example 5 Chapter 7.4

$tx'' + x' + tx = 0$ ⟶ same system as ⟶ $x^2y'' + xy' + x^2y = 0$

In Transform SPACE

We will use FROBENIUS FOR THIS TODAY

$X' = -\frac{sX}{(s^2+1)}$ superable ode

we find

$X = \frac{c}{\sqrt{s^2+1}}$

We know this transform because we did

and then as if by magic!

$x(t) = J_0(t)$ Bessels Function.

<u>Ex</u> Bessels Functions of Order 0 — Bessels Functions are some of the most important in applied math.

$x^2y'' + xy' + x^2y = 0$

$P(x) = \frac{1}{x}$ and $Q(x) = 1$

remember $Q(x) = \frac{q(x)}{x^2}$

so if $Q = 1$ then $q = x^2$

so $p(x) = 1$ and $q(x) = x^2$

or

$q(x) = 0 + 0 + x^2 + 0$

$p_0 = 1$

$q_0 = 0$

INDICINAL EQN $r(r-1) + r = r^2 = 0$ $r = 0$

so there is only one FROBENIUS SOLN we will only get one soln out of this and would need to generate another soln using other methods.

Chapter 8.4 Example 3.

lets find the one Frobenius Soln

$$y = x^0 \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^n$$ just a power series!

One case — just plug in...

$$x^2 \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x \sum_{n=1}^{\infty} c_n n x^{n-1} + x^2 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1) x^n + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

these can start @ zero

SHIFT $n \to n-2$

$$\sum_{n=0}^{\infty} c_n [n(n-1) + n] x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0$$

$$\sum_{n=0}^{\infty} c_n n^2 x^n + \sum_{n=2}^{\infty} c_{n-2} x^n = 0$$

$n=0$ $\quad c_0 (0)^2 = 0$ $\quad$ c_0 is undefined but survives.

$n=1$ $\quad c_1 (1) = 0$ $\quad$ $c_1 = 0$ we will have only even terms

For $n \geq 2$ $\quad c_n n^2 + c_{n-2} = 0$ $\quad$ or $\quad c_n = \frac{-c_{n-2}}{n^2}$

$n=2$ $\quad c_2 = -\frac{c_0}{2^2}$

$n=4$ $\quad c_4 = -\frac{c_2}{4^2} = \frac{c_0}{2^2 \cdot 4^2}$

$n=6$ $\quad c_6 = -\frac{c_4}{6^2} = -\frac{c_0}{2^2 \cdot 4^2 \cdot 6^2}$

$$c_{2n} = \frac{(-1)^n c_0}{2^2 \cdot 4^2 \cdot \cdots (2n)^2}$$

Factor out 2^2 from every term

OR

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} (n!)^2}$$

so

$$y(x) = c_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}}_{\text{This special function has a name}}$$

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$ is BESSEL'S FUNCTION OF ORDER ZERO

properties $\quad J_0(0) = 1$

and we can plot it!