Differential Equations - Notes

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 3
Now that we can solve differential equations by integration and plot slope fields for first order equations, we will start to consider more complicated cases. But, before we do this, we will investigate the idea of whether or not solutions even exist for a given differential equation and if they are unique. This is important because why bang your head against the desk trying to solve an ODE that has no solution!? Then we will learn a really nice solution method that works on both linear and nonlinear first order equations.

Existence and Uniqueness

Given and ODE, how can we know whether or not a solution exists. If we can find a solution, how can we know that it is the only one? Let's look at a few examples to motivate why this is important.

EXAMPLE:
Solve the ODE

$$\frac{dy}{dx} = \frac{1}{x} \; y(0) = 0$$

This is a First order, linear, ODE of the form $\frac{dy}{dx} = f(x)$ so we can solve it just by integrating like we did yesterday.

$$y(x) = \int \frac{1}{x}\, dx = \ln|x| + c$$

So we are able to find a general solution. Now solve for the particular solution:

$$y(0) = 0 \to 0 = \ln|0| + c$$

What is the problem here? That right! the natural log is undefined at zero! This means there is no solution to this ODE with the given initial conditions. How could we have noticed a problem right from the beginning? HINT: Look at the ode and the initial condition, see any issues?

EXAMPLE:
Consider and example from Torcelli's Law, which models water leaking from a bucket with a hole in it. The differential equation tells us how the height of the water in the bucket changes as water leaks out and the initial condition tells us how much water was in the bucket to start. For our example

$$\frac{dy}{dt} = c\sqrt{y} \; y(0) = 0$$

This is a first order, nonlinear, ODE which we will learn how to solve next week! It has more than one working solution:

$$y(t) = \left(\frac{ct}{2}\right)^2 \text{ and } y(t) = 0$$

Plug these in and test that they satisfy both the ODE and the initial condition. So which one is the right solution? BOTH are technically correct. The issue comes with our initial condition! Here we are saying that we walked up to the bucket and saw no water in the bucket leaving us with two possible conclusions. Either the water already leaked out before we got there OR there was never any water in the bucket to begin with. Our initial condition fails to give us enough information!

In general, for physical systems, when no solutions exist that means that your equations or initial conditions were probably set up wrong. When you have non-unique solutions this can be physically realistic, but you might want to check how you set up your problem.

We can actually tell, without solving the ODE, whether or not solutions exist and are unique. We use Theorem 1 on p.23 to help us with this.

Theorem 1 *Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle R in the xy-plane that contains the point (a, b) in its interior. Then, for some open interval I containing the point a, the initial value problem*

$$\frac{dy}{dx} = f(x, y),\; y(a) = b$$

has one and only one solution that is defined on the interval I.

For those of you who have had Real Analysis, this should make sense and you can read about proofs for this theorem in the Appendix of our book! For everyone else, let's unpack what all of this means in more informal language.

- We can actually show that as long as $f(x,y)$ is defined in a range around the initial condition, we know that at least one solution will exist.
- If both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are defined in a range around the initial condition, then we are guaranteed a unique solution to the ODE.
- The range over which we might expect solutions to exist and be unique is called the interval of validity. We can find this interval by drawing a number line and marking all the locations in x where the $f(x,y)$ are undefined. Then mark on that line where your initial condition is defined. Draw an interval that includes your initial condition but does not contain any discontinuities.

EXAMPLE:
Should we expect a unique solutions to the ode:

$$\frac{dy}{dx} = \frac{1}{x},\ y(0) = 0$$

Here we need to check what the function $f(x,y)$ on the RHS does at, and near, the initial condition. Plugging in $x = 0$ and $y = 0$ we see that the RHS is undefined. This means that there is no solution to the ODE. **solutions do not exist**. If instead the initial condition had been $y(3) = 0$ then we see the RHS is defined and we would at least expect a solution where the interval of validity is $(0,\infty)$ which implies that if we are looking "backward" we would need to be careful of this discontinuity at $x = 0$

EXAMPLE:
Should we expect a unique solutions to the ode:

$$\frac{dy}{dx} = \sqrt{x},\ y(0) = 0$$

First we check that the RHS, $f(x,y)$, is continuous at the initial condition $x = 0$ and $y = 0$, which it is! This tells me that we can expect at least one solution, but maybe more than one. Next we take the partial derivative with respect to the dependent variable:

$$\frac{\partial}{\partial y}(\sqrt{y}) = \frac{1}{2\sqrt{y}}$$

Now consider the initial condition $x = 0$ and $y = 0$. Here we see that the partial derivative has a discontinuity at the initial condition, so we are not guaranteed unique solutions. **Solutions exist but are non-unique**.

EXAMPLE:
Should we expect a unique solutions to the ode:

$$p\frac{dp}{dt} = t - 1,\ p(0) = 1$$

First you should rewrite it in standard form with all the variables on the RHS and only the derivative on the LHS.

$$\frac{dp}{dt} = \frac{t-1}{p}$$

Now consider the initial condition $t = 0$ and $p = 1$. When plugging into $f(t,p)$ we see that there should exist at least one solution. Now considering the partial derivative with respect to the dependent variable

$$\frac{\partial}{\partial p}\left(\frac{t-1}{p}\right) = \frac{-(t-1)}{p^2}$$

Here we see that for the given initial condition we have no issues with discontinuity so we would expect **a unique solution** to this ODE. Here the interval of validity would be $(0,\infty)$ and we would need to be careful of solutions as they approach $p = 0$

EXAMPLE:
In the above example

$$\frac{dp}{dt} = \frac{t-1}{p}$$

what if our initial condition were $y(1) = 0$? In this case we need to be more careful and instead consider limit at both y and t go to the initial condition. In the case of existence

$$\lim_{t\to 1\ p\to 0} \frac{t-1}{p}$$

because they are both first order they approach zero at the same rate and this limit exists, so we would expect at least one solution. However if we consider the limit of the partial derivative

$$\lim_{t\to 1\ p\to 0} \frac{-(t-1)}{p^2}$$

this is undefined, or infinite, because the denominator goes to zero faster than the numerator. Thus our solutions will be non-unique.

Separable First Order ODEs

We can use separation on any first order ODE of the form

$$\frac{dy}{dx} = f(x)\cdot g(y)$$

Sometimes it takes some algebra to get your solution into this form, but if you can, then separation will work. This is a method that you may have seen in Calc II. Here is the general idea

1. Classify the ODE!
2. Gather variables on opposite sides:

$$\frac{dy}{g(y)} = f(x)dx$$

3. Integrate both sides:

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

4. When possible, solve for y

This method works for both linear and nonlinear ODEs as long as the integrals are doable. For a highly nonlinear ODE sometimes it is impossible to write your solution as $y = f(x)$ because you cannot solve explicitly for y.

EXAMPLE:
Solve the ODE using separation:

$$\frac{dy}{dx} = x^3y + y$$

First classify... this ODE is first order, linear, and it can be written in separation form.

$$\frac{dy}{dx} = y(x^3 + 1)$$

Let's do separation:

$$\int \frac{dy}{y} = \int (x^3 + 1)dx$$

$$\ln|y| = \frac{1}{4}x^4 + x + c$$

$$y(x) = e^{\frac{1}{4}x^4+x+c} = e^{\frac{1}{4}x^4+x}e^c = Ae^{\frac{1}{4}x^4+x}$$

So our solution is $y(x) = Ae^{\frac{1}{4}x^4+x}$. Is this a general or particular solution? How do you know? NOTE: We will do that trick with pulling the constant out of the exponent all the time in this class!

EXAMPLE:
Solve the ODE using separation:

$$\frac{dy}{dx} = x^3y + x$$

Sneaky professor! This is a trick! It looks A LOT like the previous example but this equation is not separable! Here there is not way to separate out the y and x variables on the RHS into two separate but multiplied functions.

YOU TRY:

1. Check existence and uniqueness. If solutions exist, solve by separation:

   $$\frac{dy}{dx} = -6xy, \; y(0) = 7$$

   Helpful hint: I usually leave any constants or negative signs on the RHS.

2. Are there any values of the initial condition where we would expect problems with existence? If solutions exist for the given initial condition then solve by separation:

   $$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}, \; y(0) = 1$$

3. Solve by separation

   $$\frac{dy}{dx} + 2xy = 0$$

You can always check your answers [1] by substitution.

`The Moral(s) of the Story`

- It is always worth it to glance at existence and uniqueness! In the real world this not only saves you a lot of trouble of seeking a solution when it does not exist, but I can also highlight problems in a model. If you are modeling a physical situation with an ODE solutions should exist! If you find that solutions are non-unique, then either there are problems with your model or maybe, like Torcelli's Law, there is a good explanation for the non-uniqueness.
- Existence and uniqueness are highly dependent on the initial condition. If you see that your RHS could become undefined it is a good idea to include an interval of validity with your solution, since you want to remember where the solution"fails".
- Separation is a straight forward solution method as long as the integrals are doable. Alway check if your first order equation can be put in the form

  $$\frac{dy}{dx} = f(x) \cdot g(y)$$

- Separation works on nonlinear and linear equations, but you need your equations to be first order.

[1]

1. There should be no existence or uniqueness problems because $f(x, y)$ is well defined everywhere $(-\infty, \infty)$. The solution is $y(x) = 7e^{-3x^2}$.
2. We would have existence problems if the denominator goes to zero, of where $y = \pm\sqrt{\frac{5}{3}}$. The solution is $y^3 - 5y = 4x + x^2 - 4$.
3. The solution is $y = Ae^{-x^2}$. We are not giving initial conditions so our final answer is the general solution.