

Differential Equations – Notes

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Office Hours:

Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations – Day 4

We now have two basic solution methods, integration for equations of the form $\frac{dy}{dx} = f(x)$ and separation for equations of the form $\frac{dy}{dx} = f(x) \cdot g(y)$. We can also approximate solutions using a slope field and check for existence and uniqueness of solutions. Today we will check off a big solution category and solve ALL linear first order ODEs.

Linear First Order Equations

The general form for a linear first order ODE is:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q can be any function of the independent variable, x . Based on our last class, what can we say about whether or not solutions exist or are unique? Let's think about a few things:

- First if we put the equation in the correct form we get

$$\frac{dy}{dx} = -P(x)y + Q(x), y(a) = b$$

- Immediately we see no concerns with y however we need some conditions on the functions of x . What conditions do we need for existence and uniqueness?
- We would need both $P(x)$ and $Q(x)$ to be continuous on an open interval containing a in order for a solution to exist.
- Then when we take the first partial derivative of $f(x, y)$ we would need $P(x)$ to be continuous on an open interval containing a in order for solutions to be unique.
- So really we just need to check that $P(x)$ and $Q(x)$ are well behaved near our initial condition.

Last time when we solved ODEs by separation we broke apart the derivative in a weird way, just moving the dx to

the other side. But, technically we were doing the following:

$$\begin{aligned}\frac{dy}{dx} &= 2xy \\ \frac{1}{y} \frac{dy}{dx} &= 2x \\ \frac{d}{dx} \ln|y| &= 2x \\ \ln|y| &= \int 2x \, dx\end{aligned}$$

What we did was notice that if we integrate the function of y on the LHS then we can write the LHS as the derivative of that function and integrate both sides. Here we are actually exploiting the chain rule, because technically we are not allowed to just rip apart a derivative.

Now back to solving first order linear equations. (Keep in mind the trick above) First we will consider the solution method for all equations of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

1. We start by calculating the INTEGRATING FACTOR:

$$\rho(x) = e^{\int P(x) \, dx}$$

We integrate $P(x)$ and stick it in the exponent, ignoring the constant of integration. We will see in a minute why this is useful

2. We multiply both sides of the ODE by $\rho(x)$

$$\rho(x) \cdot \frac{dy}{dx} + \rho(x) \cdot P(x)y = \rho(x) \cdot Q(x)$$

- Now we exploit the chain rule and the product rule. We recognize that the LHS can be written as the derivative of a product. Another way to say this, we can pull a $\frac{d}{dx}$ out on the LHS.

$$\frac{d}{dx} [\rho(x) \cdot y(x)] = \rho(x) \cdot Q(x)$$

Try taking the derivative of the LHS... It totally works!! You would get back your original equation!

- All we need to do is integrate the equation w.r.t. to x

$$\rho(x) \cdot y(x) = \int \rho(x) \cdot Q(x) dx$$

sometimes this is the hardest part because these integrals can get messy!

- So our solution is

$$y(x) = \frac{1}{\rho(x)} \int \rho(x) \cdot Q(x) dx$$

Technically, if you are good at memorizing things, you can just memorize how to find $\rho(x)$ and memorize the final form of the solution above, and you can solve all first order linear ODEs. A word to the wise, for most it is better to understand the full solution method and use that rather than relying on memory every time.

Why does this work? Well it is all based on the chain rule, the product rule, and derivatives of e^x . Notice that in step 3 we used the fact that e^x has nice derivatives and a very specific form, to rewrite the LHS. We can actually check that this works:

$$\frac{d}{dx} [\rho(x)y(x)] = \frac{d}{dx} \left[e^{\int P(x) dx} y(x) \right]$$

take the derivative

$$\begin{aligned} &= e^{\int P(x) dx} \frac{dy}{dx} + e^{\int P(x) dx} P(x)y \\ &= \rho(x) \frac{dy}{dx} + \rho(x)P(x)y \end{aligned}$$

It works! That is our original LHS multiplied by $\rho(x)$. Now for lots of examples:

EXAMPLE:

$$\frac{dy}{dx} - y = e^{-x/3}, y(0) = -1$$

This is linear first order! Remember we don't care if the independent variable is inside a function, we only need the dependent variable and its derivatives to be linear for an equation to be linear. We will solve this using an integrating factor. Here $P(x) = -1$ and $Q(x) = e^{-x/3}$

- Find the integrating factor:

$$\rho(x) = e^{\int P(x) dx} = e^{-\int dx} = e^{-x}$$

- Multiply the ODE by $\rho(x)$

$$e^{-x} \frac{dy}{dx} - e^{-x} y = e^{-x} e^{-x/3}$$

- Rewrite the LHS as the derivative of a product

$$\frac{d}{dx} e^{-x} y = e^{-4x/3}$$

NOTE: You don't have to figure out the LHS, this is always $\frac{d}{dx} [\rho(x)y]$. Just plug in whatever you found for $\rho(x)$

- Integrate and solve for y :

$$e^{-x} y = \int e^{-4x/3} dx = -\frac{3}{4} e^{-4x/3} + c$$

or

$$y(x) = -\frac{3}{4} e^{-x/3} + ce^x$$

- If you are given an initial condition, apply it now

$$y(0) = -\frac{3}{4} + c = -1 \rightarrow c = \frac{-1}{4}$$

so

$$y(x) = -\frac{3}{4} e^{-x/3} - \frac{1}{4} e^x$$

You will always be able to solve explicitly for $y(x)$ in linear equations, so please do that.

EXAMPLE:

$$(x^2 + 1) \frac{dy}{dx} + 3xy = 6x,$$

First we must write it in the correct form.

$$\frac{dy}{dx} + \frac{3x}{(x^2 + 1)} y = \frac{6x}{(x^2 + 1)}$$

This is important because you need to identify P and Q based on the correct linear form. Here $P(x) = \frac{3x}{(x^2 + 1)}$ and $Q(x) = \frac{6x}{(x^2 + 1)}$

- Find the integrating factor:

$$\rho(x) = e^{\int P(x) dx} = e^{\int \frac{3x}{(x^2 + 1)} dx}$$

You should find that $\int \frac{3x}{(x^2 + 1)} dx = \frac{3}{2} \ln|x^2 + 1| + c$ but remember that for the integrating factor we can leave off the $+c$ because it would cancel out on all sides of the ODE if we left it in. So

$$\rho(x) = e^{\frac{3}{2} \ln|x^2 + 1|} = (x^2 + 1)^{3/2}$$

NOTE: Now is a good time to review rules of logs and exponents if this algebra was a surprise!

2. Multiply the ODE by $\rho(x)$

$$(x^2+1)^{3/2} \frac{dy}{dx} + (x^2+1)^{3/2} \frac{3x}{(x^2+1)} y = (x^2+1)^{3/2} \frac{6x}{(x^2+1)}$$

3. Rewrite the LHS as the derivative of a product

$$\frac{d}{dx} (x^2+1)^{3/2} y = (x^2+1)^{3/2} \frac{6x}{(x^2+1)}$$

NOTE: This is always $\frac{d}{dx} [\rho(x)y]$. Just plug in whatever you found for $\rho(x)$

4. Integrate and solve for y:

$$\begin{aligned} (x^2+1)^{3/2} y &= \int (x^2+1)^{3/2} \frac{6x}{(x^2+1)} dx \\ &= \int 6x \sqrt{x^2+1} dx = 2(x^2+1)^{3/2} + c \end{aligned}$$

or solving for y we get

$$y(x) = 2 + c(x^2+1)^{-3/2}$$

5. If you are given an initial condition, apply it now... we were not given an initial condition so our answer is the general solution.

YOU TRY:

1.

$$y' + y = 2, y(0) = 0$$

2.

$$3xy' + y = 12x$$

Answers to the YOU TRY problems are printed below ¹

¹Answers:

1. $y(x) = 2(1 - e^{-x})$

2. $y(x) = 3x + cx^{-\frac{1}{3}}$