Differential Equations - Notes

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Office Hours:
Please remember to check the class website for office hours, homework assignments, and other helpful information.

Ordinary Differential Equations - Day 5

We will continue down our path of finding new solution methods for all different types of first order ODEs. Now is a good time to start keeping a flowchart or a cheat sheet to help remind you of all of the solution methods.

Linear First Order Equations

Let's do one more example of Linear First Order Equations.
EXAMPLE:

$$x\frac{dy}{dx} + 5y = 7x^2$$

We will solve this using an integrating factor. Here $P(x) = \frac{5}{x}$ and $Q(x) = 7x$, remember that you need to put it in the correct linear first order form. Also we will note that this ODE could have existence and uniqueness problems with $x = 0$.

1. Find the integrating factor:

$$\rho(x) = e^{\int P(x)dx} = e^{\int \frac{5}{x}dx} = e^{5\ln|x|} = x^5$$

2. Multiply the ODE by $\rho(x)$

$$x^5\frac{dy}{dx} + 5x^4y = 7x^6$$

3. Rewrite the LHS as the derivative of a product

$$\frac{d}{dx}\left[x^5y\right] = 7x^6$$

NOTE: You don't have to figure out the LHS, this is always $\frac{d}{dx}\left[\rho(x)y\right]$. Just plug in whatever you found for $\rho(x)$

4. Integrate and solve for y:

$$x^5y = \int 7x^6\ dx = x^7 + c$$

or

$$y(x) = x^2 + cx^{-5}$$

5. If you are given an initial condition, apply it now.

6. Also remember that there are lots of ways to check your work. You can plug the solution back into the ODE or you can check your integrating factor by taking the derivative in step 3.

Substitution Methods

This method is similar to substitution in integration. You see an ODE that has functions inside functions and think.... "wouldn't it be great if I could get rid of that mess?" so you substitute out the mess and end up with a much simpler equation, hopefully something that is separable! Sometimes it does take some algebra to figure out the substitution. We will just show some examples of this method at work.

General Substitutions

When you see a simple algebraic expression inside a function this is a good indication that you want to do a general substitution.

EXAMPLE:
Consider the ODE

$$\frac{dy}{dx} = (x + y + 3)^2$$

Classifying, this equation is first order and nonlinear. Here we see we can't separate it and it contains both x and y on the RHS. So all is lost, pack up your bags! Right? No! Let's see if we can get rid of the messy function on the RHS using a substitution.

1. First we choose a substitution by looking at the ode. Here we see the "mess" inside the square, so let's try to substitute that out!

$$\text{let}\ \ v = x + y + 3$$

2. We can solve this substitution for the dependent variable:

$$y = v - x - 3$$

3. Next we know that we need something to sub in for $\frac{dy}{dx}$ so lets take a derivative w.r.t. x:
$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$
remember here that v is a function of x.

4. Now we can sub everything into our ODE to get:
$$\frac{dv}{dx} - 1 = v^2$$
or
$$\frac{dv}{dx} = v^2 + 1$$
and this ODE is separable!

We can solve this equation using the separation method:
$$\frac{dv}{dx} = v^2 + 1 \rightarrow \frac{dv}{v^2+1} = 1dx$$
Lets start with the integral on the LHS:
$$\int \frac{dv}{v^2+1} \quad \text{Trig Substitution}$$
$$\text{let } \ v = \tan(\theta) \ \text{ then } \ dx = \frac{1}{\cos^2(\theta)}d\theta$$
$$\text{also } \ \cos(\theta) = \frac{1}{\sqrt{v^2+1}}$$
$$\int \cos^2(\theta)\frac{1}{\cos^2(\theta)}\,d\theta = \theta + c = \arctan(v) + c$$
Now back to our ODE, we can sub the solution to the integral into the LHS, easily integrate the RHS, and finally get back to our original variables to get:
$$\arctan(v) = x + c \rightarrow \arctan(x+y+3) = x + c$$
In this case we can solve for y so our final answer is
$$y = \tan(x+c) - x - 3$$

In general any equation of the form
$$\frac{dy}{dx} = f(ax+by+c)$$
can be solved in this way. What we look for is the combination of variables inside the function.
YOU TRY:
$$\frac{dy}{dx} = \sqrt{x+y+4}$$
Answer given below [1]

[1] $2(1+\sqrt{x+y+4}) - \ln|1+\sqrt{x+y+4}| = x$

Homogeneous Substitutions

These are a special case of general substitutions and sometimes it take a bit of algebra to see that you can make a homogeneous substitution. This works for equations of the form:
$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

EXAMPLE:
$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$
When we first see this equation a simple substitution does not stand out, however if we divide by $2xy$ we start to see a pattern.
$$\frac{dy}{dx} = \frac{4x^2+3y^2}{2xy} = 2\left(\frac{x}{y}\right) + \frac{3}{2}\left(\frac{y}{x}\right)$$
now we can see the homogeneous form! We can use the substitution $v = \frac{y}{x}$.
$$\text{let } v = \frac{y}{x} \ \text{ then } \ y = vx$$
$$\text{taking the derivative: } \ \frac{dy}{dx} = x\frac{dv}{dx} + v$$
$$\text{which gives:} x\frac{dv}{dx} + v = \frac{2}{v} + \frac{3}{2}v$$
$$\text{or} \frac{dv}{dx} = \left(\frac{1}{x}\right)\left(\frac{4+v^2}{2v}\right)$$
This has reduces our problem to a separable equation. So lets use separation to solve the newly transformed ODE.
$$\frac{2v}{4+v^2}\,dv = \frac{1}{x}\,dx$$
$$\int \frac{2v}{4+v^2}\,dv = \int \frac{1}{x}\,dx$$
$$\ln|4+v^2| = \ln|x| + c$$
$$4 + v^2 = Ax \rightarrow 4 + \left(\frac{y}{x}\right)^2 = Ax$$
$$y^2 = Ax^3 - 4x^2$$
Here we used a u-sub to integrate the LHS and then since we had the natural log on both sides we took the exponential of both sides to simplify.

Now lets do a bunch of practice to test how we are doing with solving first order equations:

YOU TRY:

Solve using a homogeneous substitution:

$$xy^2y' = x^3 + y^3,\ y(1) = 1$$

Answer [2]

Solve by first choosing a substitution and then transforming the equation:

$$(x+y)y' = x - y$$

Answer [3]

$$xy' = y + 2\sqrt{xy}$$

Answer [4]

First choose a solution method that will work, then solve the ODE. Comment on if there could be any existence or uniqueness issues.

$$(1-x^2)\frac{dy}{dx} = 2y$$

Answer [5]

$$xy' + (2x-3)y = 4x^4$$

Answer [6]

[2] $\frac{1}{3}\left(\frac{y}{x}\right)^3 = \ln|x| + \frac{1}{3}$
[3] $(x+y)^2 = 2x^2 + c$
[4] $y = x(c + \ln|x|)^2$
[5] $y = \frac{c(1+x)}{(1-x)}$
[6] $y = x^3(2 + ce^{-2x})$